

R E P O R T R E S U M E S

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RESEARCH WITH A PROGRAM ON GEOMETRIC INEQUALITIES.

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REPORT NUMBER NDEA-VIIA-968-1

PUB DATE 26 JUN 64

REPORT NUMBER TM-895/102/00

GRANT OEG-7-14-0000-181

EDRS PRICE MF-\$0.18 HC-\$3.40 85P.

DESCRIPTORS- *PROGRAMED INSTRUCTION, *GEOMETRY, EDUCATIONAL RESEARCH, PROGRAMED UNITS, PROGRAM IMPROVEMENT, *LEARNING PROCESSES, *INDIVIDUAL STUDY, HIGH SCHOOL STUDENTS, PROGRAM DESIGN, INDIVIDUAL DIFFERENCES, LEARNING DIFFICULTIES, TUTORING, *AUTOINSTRUCTIONAL AIDS, SANTA MONICA, CALIFORNIA

THIS REPORT DESCRIBES THE DEVELOPMENT OF A PROGRAM IN PLANE GEOMETRY AND THE USE OF THIS PROGRAM IN EXPLORATORY RESEARCH TO GENERATE HYPOTHESES ABOUT METHODS OF PROGRAM IMPROVEMENT. AN INDUCTIVE APPROACH, BASED ON AN INTENSIVE ANALYSIS OF THE BEHAVIOR OF INDIVIDUAL STUDENTS IN A SELF-STUDY SITUATION, WAS USED TO DEVELOP PRINCIPLES OF PROGRAM DESIGN AND IMPROVEMENT. A COMMERCIALY PRODUCED PROGRAM ON GEOMETRIC INEQUALITIES WAS GIVEN TO HIGH-SCHOOL STUDENTS IN A SERIES OF TUTORIAL SESSIONS. AS EACH STUDENT STUDIED THE MATERIALS, THE EXPERIMENTER/TUTOR MODIFIED THE PROGRAM IN VARIOUS WAYS DESIGNED TO HELP THE STUDENT OVERCOME ANY LEARNING DIFFICULTIES THAT HE ENCOUNTERED. SUCCESSIVE REVISIONS OF THE INITIAL PROGRAM WERE PRODUCED UNTIL FORMAL EXPERIMENTAL COMPARISONS SHOWED THAT THE REVISED PROGRAM WAS BOTH STATISTICALLY AND PRACTICALLY SUPERIOR TO THE ORIGINAL VERSION. DETAILED RECORDS WERE KEPT OF THE PROGRAM MODIFICATIONS THAT APPEARED MOST EFFECTIVE IN IMPROVING THE INSTRUCTION. THESE EMPIRICAL RECORDS WERE THEN USED TO DERIVE SEVERAL GENERAL PRINCIPLES OF PROGRAM DESIGN. (WB)

U. S. DEPARTMENT OF HEALTH, EDUCATION AND WELFARE
Office of Education

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ED 50 3208

TECHNICAL MEMORANDUM

(TM Series)

The research reported herein was conducted under SDC's independent research program and a grant from the Office of Education, U. S. Department of Health, Education, and Welfare.

Research with a Program
on Geometric Inequalities

by

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Title VII of P.L. 85-864
National Defense Education Act of 1958
Grant 7-14-0000-181

26 June 1964

SYSTEM

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ACKNOWLEDGEMENTS

The author gratefully acknowledges the contributions of the following individuals, whose help in the development and administration of the programmed geometry materials made this study possible:

Mr. James Caballero, Santa Monica High School

Mr. Ralph Emch, Santa Monica High School

Dr. Carleton Shay, California State College at Los Angeles

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26 June 1964

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INTRODUCTION

This report describes the development of a program in plane geometry and the use of this program in exploratory research to generate hypotheses about methods of program improvement. The work was conducted as part of a two-year research project supported by the Office of Education, U. S. Department of Health, Education and Welfare.

An inductive approach, based on an intensive analysis of the behavior of individual students in a self-study situation, was used to develop principles of program design and improvement. A commercially produced program on geometric inequalities was given to high-school students in a series of individual tutorial sessions. As each student studied the materials, the experimenter/tutor modified the program in various ways designed to help the student overcome any learning difficulties that he encountered. Successive revisions of the initial program were produced in this way until formal experimental comparisons showed that the revised program was both statistically and practically superior to the original version.

The experimenter/tutor kept detailed records of the program modifications that appeared most effective in improving the instruction. These empirical records were then used to induce several general principles of program design. This approach contrasts sharply with the more conventional procedure which starts with a priori hypotheses about learning variables and then tests the hypotheses empirically.

The geometry research was conducted in eight major phases. These phases are described below, in chronological order.

Summer
1962

1. A sequence of items from a commercially produced program on plane geometry was selected. The sequence covered geometric inequalities.
2. The items were analyzed to determine program objectives, and a criterion test covering these objectives was prepared.
3. Items that appeared redundant were eliminated from the initial sequence, and the initial and shortened program versions were compared. No difference in criterion performance was found, so the short form was adopted for further study.

Fall
1962

4. Tutorial sessions resulted in a series of further program revisions.

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Winter
1962-63

5. The revised program was compared with the original, and was found to produce significantly higher criterion performance.

Spring-Fall
1963

6. Despite the superiority of the revised program, too many subjects were still unable to formulate proofs to geometry theorems. Critical subskills in the desired theorem-proving behavior were analyzed, and a new set of programmed materials was developed to teach those subskills. New criterion items were also prepared on the theorem-proving subskills.

Winter-Spring
1964

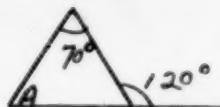
7. Tutorial sessions resulted in a series of revisions to the new materials, and confirmed their effectiveness in teaching the subskills.
8. Principles of program design and improvement were induced from the empirical observations.

SELECTION OF INITIAL SEQUENCE

A sequence of 248 items on geometric inequalities was selected for the initial program. This sequence was part of a commercially produced program on plane geometry, designed for high-school students at the tenth- and eleventh-grade levels.

Characteristics of Geometry as a Research Vehicle

Geometry has several characteristics that make it particularly useful as a vehicle for studying human learning. First, it requires the subject to use both verbal and symbolic skills. A geometry problem may be presented verbally, and may require verbal responses by the subject. (For example, he may be required to complete the statement, "An exterior angle of a triangle is equal to the sum of the two _____ angles.") Or the same problem may be presented in a form that is partly verbal and partly symbolic. For example, the problem may say: "Find interior angle A in the figure below:"



The subject must learn to be fluent in the use of both verbal and symbolic materials, and must be able to translate rapidly from one form to another. He may, for instance, be told verbally that "a" is greater than "b" and that "c" is greater than "d." He may be required to translate these statements into symbolic form (i.e., $a > b$; $c > d$), to add the two statements by manipulation of the symbols $[(a > b) + (c > d) = (a + c) > (b + d)]$, and to express the resulting relationship verbally ("the sum of a and c is greater than the sum of b and d").

The subject must not only be able to comprehend and apply information presented in either verbal or symbolic form, but must also be able to integrate the two types of information. In formulating a proof to a geometric theorem, for example, he may be given some information in a verbal statement of the problem, but he may be unable to solve the problem until he makes use of spatial relationships that can best be seen by examination of the geometric figures involved.

A second useful characteristic of geometry as a research vehicle is that it includes skills of varying complexity. The subject must learn discrete facts (e.g., an obtuse triangle is a triangle with one angle greater than 90 degrees) and simple skills (drawing a perpendicular bisector), but he must also acquire complex skills (e.g., theorem proving) that require a synthesis or integration of many component facts and skills. Similarly, he must learn both rule-following behavior and the application of strategies requiring a large degree of generalization and transfer.

Finally, geometry is a highly structured content area, with a logical sequential development from individual facts and subskills to more complex skills based on the earlier materials.

Geometric Inequalities

In schools from which the students who participated in this experiment were drawn, the subject of inequalities is presented in the second semester of plane geometry. Normally, only three or four class periods are devoted to this topic.

This subject was selected for research because (1) the definitions, axioms, and theorems presented were believed to be relatively independent of material presented in other parts of the course, (2) it was believed that regular classroom instruction would already have given these second-semester students certain fundamental skills (e.g., simple geometric definitions and basic problem-solving techniques) that would enable them to progress rapidly to more complex problems involving inequality relationships, and (3) the subject of inequalities was considered sufficiently restricted in scope so that it could conveniently be taught in a few hours of programmed instruction.

Topics Covered in Initial Program

The initial program on geometric inequalities covered the following topics:

- (1) Basic properties of distances between pairs of points (measurement procedures).
- (2) Postulate: A straight line is the shortest distance between two points.

- (3) Betweenness:

For line segments; of three points on a line, one must lie between the other two.

For angles; of any three concurrent lines, one must lie between the other two.

- (4) Use of symbols to represent inequality relationships:

$a > b$ means a is greater than b
 $c < d$ means c is smaller than d
 $e \neq f$ means e is unequal to f

- (5) Axiom: The whole is equal to the sum of its parts and is greater than any one part.
- (6) Trichotomy Axiom: A given number, when compared to another, is greater than, equal to, or less than the other number.

- (7) Arithmetic Axioms of Inequality:
- $\neq$ plus $\neq$ gives $\neq$ of the same order
 - $\neq$ plus $=$ gives $\neq$ of the same order
 - $\neq$ minus $=$ gives $\neq$ of the same order
 - $\neq$ times positive $=$ gives $\neq$ of the same order
 - $\neq$ divided by positive $=$ gives $\neq$ of the same order
 - $=$ minus $\neq$ gives $\neq$ of opposite order
- (8) Transitive Axiom of Inequalities: If one quantity is greater than a second, and the second is greater than a third, then the first is greater than the third.
- (9) Definition of interior and exterior angles of a triangle.
- (10) Theorem: An exterior angle of a triangle equals the sum of the remote interior angles and is greater than either remote interior angle.
- (11) Theorem: If two sides of a triangle are unequal, then the larger angle is opposite the larger side.
- (12) Theorem: If two angles of a triangle are unequal, the larger side is opposite the larger angle.
- (13) Theorem: In a right triangle, the hypotenuse is greater than either leg.
- (14) Theorem: The perpendicular is the shortest line segment from a given point to a given line.

Program Format

The original inequality sequence was printed in a loose-leaf binder with several items arranged vertically on each page of the program. Almost every item required the student to complete a statement by writing a few words or symbols on a separate answer sheet. After each response, the student moved a cardboard mask down the page to reveal the correct answer.

Because the inequality sequence was used out of its normal context as part of a larger program, two pages were added at the beginning of the sequence to instruct the subject on the use of the materials. Minor changes were made in the wording of five other items, primarily to delete inappropriate references to earlier topics covered in parts of the program not seen by the subjects. In addition, all items were reproduced one to a page, to simplify subsequent program revisions. In the remainder of this report, the term "initial program" refers to the original inequality sequence with the modifications described in this paragraph.

ANALYSIS OF PROGRAM OBJECTIVES

AND PREPARATION OF CRITERION MATERIALS

The inequality sequence was analyzed to determine the major learning objectives, so that a criterion test covering these objectives could be prepared. For each of the program topics listed above, three levels of conceptualization and transfer were identified:

1. The first level (for convenience, labeled level "A") required the subject to learn specific responses to specific stimuli presented in the program (e.g., an item requiring him to fill in the missing words of a definition or axiom). This objective required minimal generalization and transfer.
2. A second performance level (level "B"), involving somewhat greater transfer, required the subject to apply a definition, axiom, or theorem to a particular problem. For example, he might be shown a triangle with two unequal angles, and asked to identify the larger of the two opposite sides using one of the inequality theorems. Such responses are generally interpreted to demonstrate "understanding" of principles, as contrasted with mere memorization of rules.
3. The third performance level (level "C"), required the subject to synthesize response elements (i.e., individual definitions, axioms, and theorems) in new patterns and combinations to solve novel multi-step geometry problems. The major example of this class of behavior is the formulation of proofs to theorems that the subject has not previously seen.

A pool of over 200 criterion items was written. These items were labeled by topic and were also classified by performance level according to the three-way classification system described in the preceding paragraphs.

A post-training criterion test was written using a sample of items from the criterion pool. Items were sampled to ensure that each major topic was covered by at least one item, and an attempt was made to include roughly equal numbers of items at levels A, B, and C. Two judges independently classified each item, and items on which the judges disagreed were eliminated from the criterion test. The test contained 50 items, including 41 multiple-choice and 9 constructed-response questions. All of the constructed-response questions had multi-step solutions, bringing the total possible score to 65. Possible scores on A, B, and C items were 21, 20, and 24 respectively.

Appendix A shows the criterion test with items marked A, B, or C according to the performance level judged to be required.

FIRST PROGRAM REVISION: ELIMINATION OF REDUNDANT ITEMS

The initial inequality sequence was given on an exploratory basis to two high-school students, primarily to refine the procedures used in administering the tutorial sessions. It was observed that a sizeable number of the items seemed redundant; frequently, practice items on a topic appeared in the program considerably beyond the point at which the subject had demonstrated mastery of the topic. Moreover, most of the apparently superfluous items were on the earlier and simpler topics, with relatively little repetition or explication of later, more complex topics. Consequently, a revised sequence was prepared which contained only 180 of the original 248 items.

An experiment was next conducted to determine whether program effectiveness had been reduced by elimination of the "redundant" items. Thirty students in a second-semester geometry class at Culver City High School were used as subjects. None of the subjects had studied inequalities previously. Because it was expected that the short program would teach as well as the long form, and because the experimenters wished to obtain as much response data as possible on the short program, 20 subjects were given the short form and 10 the long form. Within this limitation, the subjects were randomly assigned to the two groups. The programs were given to all subjects during their regular classroom sessions on five successive school days. Each subject proceeded at his own pace through the materials. Instruction was monitored by the regular classroom instructor, who answered procedural questions but not substantive questions. The subjects were not given homework assignments during the period covered by the study.

Illness prevented one subject in each group from attending classes, so 19 completed the short program and 9 the long form.

At the completion of the final study session each of the 28 subjects took the criterion test. Those who received the short program obtained a mean score of 22.2 with a standard deviation of 4.5; those receiving the long form had a mean of 23.7 and a standard deviation of 4.7. The significance test yielded a t value of 0.73, which is not statistically significant.

Because the programs were administered in the ongoing classrooms, a precise measure of each subject's study time was not possible. If each subject had brought his papers to the instructor on completion of the program, this would have distracted other subjects still working. Consequently, they were told to keep their papers, to note their completion time, and to study other materials after they finished. It was observed, however, that they frequently referred to the program after the recorded completion time. For this reason study time was not included as a dependent variable. Informally, however, it was noted that subjects receiving the short program finished consistently ahead of their classmates with the long form, as was expected.

Since no significant differences were found in criterion scores for the two groups, and since the shorter program appeared to make more efficient use of the subjects' study time, the short form was adopted for subsequent revisions.

Two further uses were made of the experimental data obtained in comparing the original and short programs. First, an index of discrimination was calculated for each criterion item, based on comparative performance of the 10 subjects making the highest and the 10 making the lowest total scores on the criterion. (The two experimental groups were pooled for this purpose.) Three items found to discriminate negatively were eliminated from the criterion. Four other items with high error counts were found to contain errors or to be ambiguous; these also were eliminated.

Another use of the experimental data was an analysis of responses to questions in the program. Items with high error counts suggested possible weaknesses in the program. Points of difficulty were thus tentatively identified in advance of the tutorial sessions, and the experimenters were able to do some preliminary planning of remedial procedures that might be employed in the tutorial sessions.

TUTORIAL SESSIONS RESULTING

IN FURTHER PROGRAM REVISIONS

Because many program items involved heavy prompting, it was felt that the questions contained in the program might be inadequate for the detection of student learning problems requiring tutorial aid. Consequently, several of the criterion-pool items not used in the post-training criterion test were added at appropriate places in the program to permit more accurate evaluation of student learning progress during the tutorial sessions.

The short form of the program was given in individual tutorial fashion to seven subjects from second-semester geometry classes in Culver City High School. Each subject studied in a small room, with the experimenter seated in the same room and with a tape machine recording all oral interactions between subject and experimenter. The program was presented to each subject in five sessions, each of which lasted from 30 to 60 minutes.

As the subject proceeded through the programmed materials, he wrote his answers on separate work sheets which provided a permanent record of his responses. The experimenter made a note of any point in the program that seemed to give difficulty, as evidenced by incorrect answers, long delays in response, facial and bodily signs of stress, and verbal statements by the subject. He would ask the subject to describe the nature of his difficulty, and to verbalize the methods by which he was attempting to solve the problem. The experimenter attempted to help the subject overcome his difficulty through various tutorial techniques. The particular techniques used by the experimenter depended on his assessment of the learning problem and on his past experience in trying to remedy such problems. If his first remedial attempt did not help, he would try other techniques until the subject was able to answer new questions on the topic, at which point the student was allowed to proceed in the program.

When the experimenter found that a tutorial technique helped overcome a particular type of learning problem in one part of the program, he would attempt to use the same technique for similar problems in other parts of the program. Likewise, a technique that proved successful for one subject would be used with subsequent subjects. Thus, tutorial procedures were iteratively modified on the basis of experience with the series of subjects. Tutorial procedures that were consistently effective were incorporated in successive program revisions as new or altered items. The final revised program contained 167 items.

The following section of this paper summarizes some of the most serious and most consistent difficulties encountered in the original program version (for convenience, abbreviated "OP"), and the changes made in the revised program version ("RP") to remedy those difficulties.

Addition of Preview Materials

OP provided almost no advance information to the subject about the topics he would be studying in the program or about the skills he would be expected to learn. As a result, many subjects appeared to lack an adequate goal-orientation, and made errors indicating confusion among different parts of the program. Three specific examples of this general problem, and of steps taken in RP to alleviate the program weaknesses, are discussed below.

- A. Problem: Inadequate warning of change in topics. OP often went from one topic to another without warning or notification to the subject. In some instances this caused him to make errors on initial items in the new topic. Even when he was able to answer questions on the new topic, he frequently slowed down and sometimes reported confusion about how the new item related to the earlier topic.

Solution. Items were added or modified in RP to notify the subject explicitly of changes in topic. (An example is presented in Appendix B.) These added instructions provided cues for interpreting and responding appropriately to the items on each new topic. They helped maintain a more regular study pace through the topics and appeared to reduce errors at transition points in the program.

- B. Problem: Inadequate statement of learning objectives. In most complex multi-step problems, such as providing a theorem, OP did not state the terminal objective of the problem until the solution was almost completed. Typically, OP began with an item describing some obvious characteristic of one element of the geometric figure concerned in the problem. Several more items expanded further on this characteristic and added information about other elements of the figure. A lengthy sequence of such items led in logical progression to the desired proof, but only in the final steps was the theorem itself made explicit.

Subjects were almost always able to answer individual questions contained in the early items, but sometimes made errors on questions in the final stages of the proof. In addition, they were frequently unable to reconstruct the proof on the criterion test.

Answers given to the experimenters' questions indicated that most subjects could see no "point" in the early items, which seemed trivial and unrelated to any significant problem. Because they could not see where the early items were leading, they tended to read them superficially, spending only long enough to give the necessary responses (which were heavily cued). In subsequent parts of the program they were not able to reproduce the earlier responses or to apply them correctly in new problems.

Solution. Items were added or rewritten in RP so that, at the beginning of a long sequence of items on a complex problem, the subject was told what the problem was. For example, one sequence in RP began with the following statement: "Let's try one of the inequality axioms on a geometric problem. If you are given two equal angles, and unequal parts of each of these angles, you can find the relative size of the remaining angles." Thus the nature of the problem was specified and the subject was told that subsequent items, trivial though they might seem individually, would lead systematically to the solution of that problem. After these revisions, fewer errors were made, particularly on the criterion test where the problem solutions had to be reproduced with minimal prompting.

- C. Problem: Inadequate preview of sequence of subsequent topics. OP proceeded from one topic to another with no advance notice to the subject about the sequence of topics that he was to receive. That is, he had no "table of contents" telling him, for instance, what the next five topics would be. This did not appear to cause difficulty when the topics were discrete and independent, but it did create learning problems when the sequence included interrelated topics. For example, the program covered six Arithmetic Axioms of Inequality, specifying different operations performed on two or more inequality statements. The first one or two axioms were generally learned with little difficulty, but the subject showed interference on the later axioms. This interference appeared to result from the fact that several of the Arithmetic Axioms were very similar in appearance but required different applications.

Verbal reports by the subjects indicated that they usually found little difficulty in answering program questions on individual topics, but that they frequently felt "lost" and could not see an over-all direction or pattern in the sequence.

Solution. "Preview" items were added to RP specifying some of the topics that followed in the program. One such item, for example, stated that in subsequent items the subject would study six different rules, or axioms, for arithmetic manipulation of inequality statements (see Appendix C). It then specified what operations the rules allowed him to perform, and cautioned him that, although the rules might look similar, each had a distinct and separate application. Subjects receiving RP showed less confusion among the six Arithmetic Axioms and were able to apply them more accurately in later parts of the program.

Another example of the use of previews occurred in RP immediately before a series of items designed to produce theorem-proving subskills. The preview stated that succeeding items would teach the subject (1) how to get a problem (theorem) down on paper simply and accurately, (2) how to use what is "given" in the theorem to generate initial steps in the proof, and (3) how to use the "prove" part of the theorem to generate other steps in the logical chain of statements constituting the required proof. As the result of the preview items, subjects made fewer errors in using the subskills to prove theorems.

Addition of Review Materials

- A. Problem: Lack of review on individual topics and sequences of topics. OP contained practice items on many of the topics, but most of these items occurred immediately after each topic was first presented. The program provided little additional review. As a result, subjects were often able to perform satisfactorily as they studied each new topic in turn, but made errors later in the program when they attempted to apply the earlier information to more complex problems. On the criterion test, moreover, they made errors even on the individual topics that they appeared to understand while studying the program.

A concentration of errors occurred at the beginning of each day's lesson. OP was organized as a single long sequence of items; it did not specify points at which the program should be broken. The instructions accompanying OP suggested that in each new study session the subject should repeat the last three or four items from the sequence studied in the preceding session, but this informal method of review proved entirely inadequate, as indicated by the response errors.

Solution. Two review procedures were added in RP. In one procedure, designed to strengthen the subjects' performance on an individual topic or skill, practice items were inserted at intervals after the initial presentation of the topic. (Appendix D shows one such item.) In the second procedure, summary items reviewing several earlier topics were added. Although the summary items provided practice on individual skills, their main purpose was to aid in conceptually organizing a

sequence of topics, that is, in relating the topics to each other and to subsequent topics in the program, and in discriminating among the different topics. Thus, the review procedures complemented the effects of preview materials in helping the subject to reduce interference among different topics in the program.

RP was divided into five lessons, each contained in a separate booklet and each designed to require approximately 40 to 60 minutes of study time. Wherever possible the break between booklets was made at the conclusion of a topic rather than in the middle. Reviews were added at the beginning and end of most lessons. The review at the end of a lesson summarized topics covered in the lesson, and possibly in earlier lessons. (Appendix E shows one such review.) These reviews were designed to improve the subject's organization and retention of the materials he had studied up to that point. The review at the start of a lesson (see Appendix F) provided a brief rehearsal of topics studied in the preceding sessions, so that the subject did not waste time trying to remember what he had covered earlier, but could attend almost immediately to new topics.

The number of review items required at the beginning and end of a lesson, and the optimal number of earlier topics covered by the review, were directly dependent on the time interval between study sessions. A review that was satisfactory for a 24-hour interval was too short when sessions were separated by a weekend. In highly structured subject areas such as geometry, it appears that almost as much consideration should be given to the planning of proper break points in the program, and of reviews associated with the break points, as to the design of the individual items within a session.

- B. Problem: Lack of review of steps within a topic. When OP presented a long sequence of interrelated items, such as those demonstrating a theorem proof, subjects responded correctly to most individual items but were often unable to answer subsequent questions dealing with relationships among the items. It appeared that they had become so engrossed in the details of the individual items that they did not attend to relational statements or to the over-all pattern of the total sequence. In short, they "could not see the forest for the trees."

Solution. Theorem proofs in RP were rewritten so that each item summarized all the information provided up to that point in the sequence and added one new piece of information. (See Appendix G for an illustration of one such sequence.) This cumulative mode of presentation allowed the subject, at each step of the proof, to see how each new step was related to earlier steps, and how it advanced the proof toward the terminal objective. Fewer errors were then made in reproducing the proof, both in the program and on the criterion test.

Addition of Materials to Fill Gaps

OP left gaps at several important points in the program. That is, it failed to provide basic information on certain points that were directly related to the program objectives. Several examples of such gaps, and of materials added in RP to fill the gaps, are described below.

- A. Problem: Lack of information restricting the application of a rule. OP presented several important rules (the Arithmetic Axioms of Inequality) that could be applied only to two inequality statements of the same order. However, the program never stated the explicit criterion by which the subject could determine when two inequality statements were in the same order. Neither did the program present enough examples in the use of the Arithmetic Axioms for him to induce this concept. As a result, numerous errors occurred, throughout the rest of the program and on the criterion test, whenever the subject attempted to solve problems that required application of the Arithmetic Axioms.

Solution. RP contained three items explicitly stating the criterion for judging whether two inequality statements were of the same order and describing a transformation procedure by which inequality statements of opposite order could be put in the same order. (See Appendix H.) A resulting decrease in errors on the program and on corresponding parts of the criterion test was observed.

- B. Problem: Lack of information generalizing the application of a rule. OP gave instruction on the addition of two inequality statements, but did not specifically indicate that the same addition procedure could be applied to three or more inequality statements. In later portions of the program, some subjects made errors in problems requiring them to add several inequalities.

Solution. RP added two items explicitly stating the addition procedure for three or more inequality statements. (See Appendix I.) These items appeared to reduce student errors on related program and criterion items.

- C. Problem: Lack of directions for pronunciation of terms. Some subjects receiving OP were unable to remember the term "Trichotomy Axiom," or to apply the label accurately. Responses to questions by the experimenter indicated that these subjects were unable to pronounce the word "trichotomy" and that this interfered with their use of the word in written form. (Note: It did not appear to be necessary for them to pronounce the word correctly, but only to have some convenient phonetic reference.)

Solution. An item was added to RP presenting the word phonetically ("Try-Kot-Oh-Me"). Subjects appeared to make fewer errors in written applications of the word as the result of the new item.

Elimination of Irrelevant and Unnecessary Materials

Parts of OP did not contribute to the program objectives (as identified in the program analysis and as represented by the criterion test) because they were irrelevant or because they provided training on skills previously acquired. Four examples of irrelevant and unnecessary materials in OP, and of the changes made in RP to remedy these defects, are described below.

- A. Problem: Item content already familiar to subject. OP gave practice in the measurement of line segments. These items taught nothing new, since the subjects were already able to measure line segments and to deal symbolically with simple concepts of measurement.

Solution. Items requiring the subject to measure line segments were either eliminated or modified so that the subject dealt with line segments for which dimensions were already provided. These changes reduced training time in the tutorial sessions.

- B. Problem: Item content irrelevant to program objectives. OP contained formal statements about certain abstract geometric properties. (For example, one item stated that "In geometry, we assume that for any pair of points in the same plane there is a number called the distance between the two points." Further in the program, it was noted that, "For any three points on a line, exactly one of them lies between the other two.") While these statements could be relevant in a program on geometric inequalities, most of them were not relevant in OP because (1) they did not add substantive information but simply confirmed the subject's intuitive knowledge on the topics concerned, and (2) the statements were isolated concessions to formalism, since there was no systematic effort in OP to teach geometry as a rigorous system of formalized definitions and operations; therefore, the teaching of formal definitions was not considered a program objective.

Solution. In RP, most statements about formal geometric properties were eliminated entirely. Formal statements that contained some information in addition to, or in conflict with, the subject's intuitive notions were retained.

- C. Problem: Irrelevant responses. OP contained numerous items requiring trite or irrelevant responses. That is, the responses called for were so obvious that they did not need further discussion, or they dealt with some notion that was not central to the main point being taught. Such items not only wasted time, but also caused some subjects to take the whole program less seriously, as indicated by their voluntary comments.

Solution. Some items requiring trite or irrelevant responses were eliminated in RP. Others were restated so that the responses called for were significant and more central to the program objectives. For example, one item in OP showed a line with three points on it and required the subject simply to list the three points. The item was altered in RP so that the subject was required to name the line segments produced by the points. While both items were relevant to the program topic, the response in the item from OP was trite, whereas that in the item from RP represented a significant step by the subject toward the program objectives.

- D. Problem: Overt responses a hindrance to integrating complex skills. OP required an overt response in almost every item. This format was found effective in teaching discrete facts and other information that could be presented in a small number of items. It was also successful for skills that were conceptually simple but that required a lengthy sequence of practice items to bring the subject to a high performance level (e.g., translating from a verbal to a symbolic statement of inequality relationships). Subjects had difficulty, however, in synthesizing information from long sequences designed to produce complex skills (e.g., proving a theorem). It appeared that their overt responses to each item slowed their progress and interrupted the integration of factual information and subskills leading to the higher-level skills. By the time the subject responded to a question about an individual step in the sequence and received feedback about the response, he had forgotten how that step was related to other steps and to the over-all solution strategy.

Solution. In portions of RP designed to teach complex skills (particularly theorem proving), the questions were deleted from some of the items. The items were written to present information as clearly and unambiguously as possible, and to allow rapid movement through the sequence of facts and subskills. (See Appendix J for one example of such items.) Subjects were not required to respond until after they had been given an opportunity to study a sequence of interrelated steps. This modification appeared to increase correct responses on the more complex skills and to make more efficient use of study time.

Amendment of Prompting Procedures

- A. Problem: Absence of prompts necessary to elicit correct responses. OP occasionally required responses for which no adequate cues or prompts were provided, either in the item containing the question or in earlier items. Such items produced a very high error rate and caused the subjects to proceed more slowly on several succeeding items. In effect, the subject learned nothing from the item, since he simply guessed at the answer; his only instruction was from the feedback message.

Solution. Items were reworded, or new items added, to prompt the subjects' responses. Figure 1, below, illustrates one such revision. Part A of the figure shows an item from OP, and Part B the revised item from RP. The sentence added to RP used the word "between" to describe a spatial relationship between points in a particular line drawing. This prompted subjects to use the same word in completing a statement about a general geometric principle.

Fig. 1

Illustration of Addition of Prompts

Part A shows an item as it appeared in the original program, and Part B as it was modified in the revised program.

A

One further postulate, that is, geometrical assumption, which we make in plane geometry is illustrated in the figure. We assume that for any three points on a line, exactly one of them lies _____ the other two.



B

One further postulate, that is, geometrical assumption, which we make in plane geometry is illustrated in the figure. Notice that point B lies between point A and point C. In general, we assume that for any three points on a line, exactly one of them lies _____ the other two.



- B. Problem: Too-rapid fading of prompts. OP often removed prompts and cues abruptly, with the result that the subject was not able to respond correctly to the reduced cues in later parts of the program. Early in the program, for example, the subject was taught how to manipulate the direction of an inequality sign using actual numbers. For instance, he learned that " $8 > 5$ " means the same thing as " $5 < 8$." The use of numbers made it simple for him to be certain that the symbol faced in the right direction. In later problems, however, he was required to manipulate the symbol using letters, and was frequently unable to do this correctly.

Solution. Prompts and cues were faded more gradually in RP. For example, immediately after the inequality symbol ($>$) was presented, the subject was given practice in manipulating the symbol with numbers. The numbers were then replaced in successive steps by letters, enabling the subject to see how the same operations could be applied with both numbers and letters. As a result, subjects performed better not only on that particular part of the program, but on later problems which required manipulation of the symbol as one step in a multi-step solution.

Addition of Branching Capabilities

Problem. OP was a fixed-sequence program with no provision for branching back to earlier items when the subject showed weakness on a particular topic. Too many subjects advanced from one topic to another without having adequately mastered each topic in turn. This was evidenced by their performance on the criterion test and by their inability to apply information presented in one part of the program to problems that occurred later in the program. In addition, some subjects voluntarily expressed a desire to repeat some sections of the program.

Solution. Optional Review Items were added at several points in RP (see Appendix K). An Optional Review Item listed some or all of the major topics covered previously in the program, and suggested that the subject should review the materials on any topic he felt he had not adequately mastered. The Optional Review Item listed the page numbers associated with each topic; the subject could repeat all the items on a topic, or simply scan through the items to check his memory of some particular point on that topic. When he had completed his review of a topic, he could return to the Optional Review Item.

The optional review capability appeared to contribute substantially to the effectiveness of RP. Most subjects voluntarily branched back to earlier items on at least one topic, and their verbal statements indicated that these reviews improved their grasp of certain topics which they had not previously understood. These statements were supported by the fact that subjects who repeated items on a topic gave a higher percentage of correct responses on those items than they had the first time through the topic.

Program flexibility was also increased by the addition of programmed branches at several points in RP (see Appendix L). A programmed branch item tested the subject's knowledge of a topic and, depending on his response, sent him ahead to a new topic or back for a review of the earlier topic.

EXPERIMENTAL COMPARISON OF ORIGINAL AND REVISED PROGRAMS

This part of the study was conducted in three stages: (1) administration of a mathematics pretest to geometry classes from which subjects were to be drawn, (2) presentation of the two forms (original and revised) of the geometry program, and (3) administration of a post-training criterion test to all subjects.

Procedures

A mathematics pretest was given to all members of three second-semester geometry classes. Students in each class were then randomly assigned to an experimental group and a control group. Each subject in the control group received the original program sequence (OP) of 248 items, while subjects in the experimental group were given the revised sequence (RP) of 167 items. Individual subjects receiving RP repeated different numbers of items depending on the branches they took in the program. Although it was not possible to record the exact number of repetitions, it is estimated that the experimental group saw, on the average, around 200 items.

Both forms of the program were administered during the regular geometry class periods, on five successive days. Subjects were not informed that others in the same room were studying a different program. Instruction was conducted in the regular classrooms, in the presence of the regular instructor. The instructor was allowed to answer procedural questions but could not give substantive information.

All subjects were told to take as much time as they needed in studying the program, but not to delay unnecessarily. They were instructed to set their programmed materials aside after they finished all the items for that day (including programmed and optional review), and to spend the remainder of the period studying some other subject area. This procedure was used to minimize distractions that might occur if subjects left the room or turned in their programs when they completed a day's sequence. Subjects were also told to note the time at which they completed each day's lesson.

The programs were collected at the end of each day, and the subjects were given no homework assignment in geometry during the five days of programmed instruction. The subjects were informed that they would be tested on the

materials covered in the program, and that the test results would be included in their final course grades. The criterion test was administered on the next school day after completion of the programs. Subjects were given as much time as they required to complete the test.

Subjects

A total of 32 experimental subjects and 32 control subjects from three second-semester geometry classes in Santa Monica High School participated in the experiment. They had not yet studied geometric inequalities in their regular classroom instruction, but had worked in the previous semester on basic geometric definitions ("angle," "isosceles triangle," "congruency," etc.) and on equality relationships. None of the subjects had participated in the earlier tutorial sessions.

Mathematics Pretest

The mathematics portion of the California Achievement Test was administered to all the subjects before they received instruction on geometric inequalities. This subtest contains 60 items on mathematics reasoning in arithmetic, algebra, and geometry, and 80 items on mathematics fundamentals.

The purpose of the pretest was to determine whether criterion performance differences (if any) between experimental and control groups could be accounted for by differences in mathematics achievement.

Criterion Test

The criterion test used in this study was the same test described on page 10, except that five negatively discriminating and non-discriminating items had been eliminated, reducing the maximum total score to 55. Of this total, 20 points were possible on part A (minimum transfer), 18 on part B (moderate transfer), and 17 on part C (maximum transfer).

Results

Pretest Scores: Pretest scores for the two groups are summarized in Table 1, below. The calculated t value, 0.61, is not significant.

Table 1

Comparison of Pretest Scores for Experimental Group* (Revised Program) and Control Group* (Original Program)

Mean Scores		Range		Standard Deviation		
Exper.	Control	Exper.	Control	Exper.	Control	
138.6	135.9	101-165	85-164	15.4	19.0	0.61(NSD)

*N=32

Criterion Scores: Results on the criterion test are summarized below in Table 2. (Raw scores are given in Appendix M.)

A significant difference in favor of the revised program was found on the total test (.01 level), and on parts A (.01 level) and B (.05 level; one-tailed test). (The part B comparison was not significant on a two-tailed test; however, the one-tailed test was considered applicable because a difference in favor of the experimental group had been predicted.) No significant difference was found on part C, the portion of the test requiring maximal transfer.

Table 2

Comparison of Criterion Scores for Experimental Group
(Revised Program) and Control Group (Original Program)

	Mean Score		Standard Deviation		t
	Exper. (N=32)	Control (N=32)	Exper.	Control	
Total Test	35.7	31.4	5.6	6.6	2.8*
Part A	17.1	14.7	2.3	2.8	3.7*
Part B	13.2	12.0	2.3	2.9	1.7**
Part C	5.4	4.7	1.7	2.2	1.2(NSD)

*Significant at .01 level

**Significant at .05 level (one-tailed test)

An attempt was made to record study times for each subject, but it was impossible to determine accurately when each day's programmed sequence had been completed. On numerous occasions, subjects were observed to review the program after they had recorded completion of the session, despite contrary directions by the experimenter. Consequently, study time was not used as a dependent variable in the data analysis. As an approximation, however, the experimental group took an average of 181 minutes for the total program, or a mean of 36 minutes per lesson, while the control group took 202 minutes for a mean of about 40 minutes per lesson. Thus the data on study time, while not subject to rigorous interpretation, appeared to favor the revised program.

PREPARATION OF NEW MATERIALS TO TEACH THEOREM PROVING

The results of the comparison between OP and RP indicated a significant improvement in most parts of the program. Subjects receiving RP were better able to reproduce definitions, axioms, and theorems and to apply them to simple geometric problems. They were also better able to reproduce proofs of theorems presented in the program. However, RP was unsuccessful in teaching subjects to prove theorems that they had not previously seen. This failure was reflected in the low scores on part C of the criterion test.

Logically, there appeared to be three possible explanations for this failure in the program:

1. The program did not adequately teach basic definitions and operations relating to geometric inequalities.
2. There were other prerequisite subskills, not related specifically to inequalities, that were not covered in the program and not available in the subjects' entry repertoire.
3. The program did not adequately teach the higher-order skill of integrating basic information and subskills to produce novel proofs.

The first of these three explanations was rejected on the basis of the satisfactory performance on parts A and B of the criterion test. A detailed analysis of responses on part C suggested that either the second or the third explanation, or both, might be correct. It was found, for example, that some subjects were unable to identify the "given" and "to prove" elements in verbal statements of the theorems, or to represent the theorem properly by drawing a figure. Others were unable to select from a list of geometric "Ideas" (e.g., definitions, axioms, and theorems) the Idea which would best provide a necessary link in a chain of statements forming a proof.

In brief, it appeared that the difficulty was not specific to the subject of inequalities, but involved a weakness in certain basic problem-solving skills that the subjects were supposed to have acquired in their first-semester work on equality relationships. It was also clear that the subjects' deficiencies could not be remedied by simply adding a few new items at various points in the existing program. Rather, an extensive effort was required to develop a new set of materials specifically designed to teach the prerequisite theorem-proving subskills. The remaining portions of this report describe the efforts to analyze theorem-proving behavior and to develop a new set of materials (called, for convenience, the "Proof Booklets") to produce the subskills that were identified.

Analysis of Theorem-Proving Subskills

Four sources of information were used to identify critical theorem-proving subskills:

1. A logical analysis of steps required in proving a theorem, and of the subskills required for each step.
2. An analysis of subjects' behavior in proving theorems in the program and in the criterion test.
3. Anecdotal descriptions, prepared by mathematics teachers, of the theorem-proving behavior of students in their geometry classes.
4. Descriptions of the decision logic used in a computer that has been programmed to prove theorems in elementary Euclidean plane geometry.^{1,2}

From the analyses it was determined that two basic types of activity, differing markedly in complexity and difficulty, were required to prove a novel geometry theorem. One type of activity involves the application of a specific subskill such as representing the theorem properly in a figure, or isolating the "given" portion of the problem statement. The second and more difficult activity is that of combining and integrating the various subskills to form the necessary links in a chain of logically connected statements; this chain, when completed, forms the theorem proof.

A further analysis of these two basic types of theorem-proving activity is presented below.

1. Component subskills

- a. Interpreting a verbal statement of the problem. The subject must be able to translate a verbal statement of a theorem into symbolic form. That is, he must draw a figure that accurately represents the problem statement and must label parts of the figure appropriately. He must also be able to identify by appropriate symbols what is given and what he is to prove.

¹H. L. Gelernter, "Realization of a Geometry Theorem-Proving Machine," Information Processing, Proceedings of the International Conference on Information Processing (Paris: UNESCO, June 1959), pp. 274-282.

²H. L. Gelernter and N. Rochester, "Intelligent Behavior in Problem-Solving Machines," IBM Journal of Research and Development (October, 1958), pp. 336-345.

- b. Finding the content of a geometric Idea. The subject must be able to strip the verbal statement of a geometric Idea (i.e., a postulate, axiom, theorem, or definition) down to its essential contents. For example, he must recognize that the basic content of the statement, "base angles of an isosceles triangle are equal," is simply "angle equals angle" (or, symbolically, " $\angle = \angle$ ").
- c. Generating a list of Ideas having a specified content. Given a particular content (e.g., " $\angle = \angle$ "), the subject must be able to generate a list of previously learned Ideas having that content. These Ideas may have been presented at any time during his earlier geometry instruction. Thus, he must have access to a "catalog" of all earlier geometry Ideas, and must select from it those Ideas having the specified content.
- d. Applying an Idea in a specified problem. Out of the list of Ideas having some specified content, the subject must be able to select those that fit the statement of a particular problem. For example, one Idea might be, "The hypotenuse of a right triangle is longer than either leg." If this Idea is to be applied in the problem, then the theorem must deal with a right triangle.

2. Procedures for combining subskills to form proofs

After the subject has acquired the theorem-proving subskills, he must have a general plan for organizing and integrating these subskills. That is, he must apply the subskills in a sequence and a direction that will produce a chain of logically connected geometric statements forming the required proof. (Later, he may use more refined techniques to reduce the amount of time and the number of steps necessary to form the proof.)

It is difficult to isolate or to classify the specific behaviors required in synthesizing a sequence of subskills to form a theorem proof. Taken as a whole, these behaviors constitute a complex problem-solving strategy. Some components of this strategy can be taught as rule-following behavior. For example, the subject can be taught always to start by drawing the figure, and then to put down the "prove" parts of the problem. Other parts of the strategy cannot be stated as explicit rules; they involve judgmental decisions based, in large measure, on the subject's experience in solving similar problems. He cannot be told, for example, the precise conditions under which

he should abandon one logical chain or sequence of geometric statements and pursue an alternative chain.* He can only be given certain guidelines, such as to look first for a type of sequence which, in the past, had led to a simple solution with few links (steps). A subject with limited experience in such problems may not, at first, be able to use this guideline very fruitfully. Furthermore, even for the more experienced subject this guideline is only a heuristic; it does not guarantee that a particular chain will lead successfully to the required proof.

Some of the rules and heuristics that have been found useful in proving geometric theorems are discussed in the following section, which describes the initial set of Proof Booklets.

Development of Proof Booklets

The initial set of new materials consisted of five Proof Booklets, one for each of the four major subskills and one for the general theorem-proving strategy. These booklets, totaling 92 items, were designed to be studied by the subjects after they completed the five booklets on geometric inequalities (that is, the RP booklets described on page 16.) The objectives and content of each Proof Booklet are summarized below.

Booklet I: Getting the problem on paper. This booklet was intended to help the subject analyze the problem (theorem), determine his starting point and final objective, and represent the problem accurately on paper using appropriate symbology.

An item falling early in the sequence told the subject that he could not hope to solve geometry problems until he could read and comprehend verbal statements about the problem. As a start, he had to identify the facts provided in the problem and determine what he was expected to do with those facts (e.g., what inequality relationship he was to prove).

*In this regard, the strategy taught to the subject is different from that programmed in a theorem-proving computer (Gelernter, 1959). All directions to the computer must be operationally specified. It may be instructed, for example, to try every possible solution chain according to some priority ordering; the computer will work with each chain until it produces the required proof or fails (according to some explicit criterion). The computer's high operating speed and perfect retention permit it to pursue each possible solution to its logical conclusion before turning to the next. By contrast, different techniques must be taught to the subject whose slower speed and imperfect memory prevent him from exploring systematically the entire complex network of possible solution paths (or even a major part of this network except for very simple problems). He must use more subtle and less operationally defined cues in deciding when to try a particular path and when to abandon it for another.

He was told that these two kinds of information might be given in many different forms, and that some forms were easier to interpret and use than others. He was told that the "if...then" form was particularly convenient, and that he should represent the information on paper in this way. For example, the statement, "The hypotenuse of a right triangle is greater than either leg," could be rewritten, "If there is the hypotenuse of a right triangle, then it is greater than either leg." In this format it was simple to identify what was given (i.e., a right triangle with hypotenuse) and what was to be proven (i.e., the hypotenuse is greater than either leg). The subject was given practice in converting various forms of problem statements into "if...then" and "given-prove" forms.

The subject was then told that after he had stated the problem in "if...then" form, he should draw a sketch representing the problem. This, he was told, would clarify the problem and allow him to see important spatial relationships more easily. He was given practice in drawing and labeling figures to correspond with verbal problem statements, and in identifying the "given" and "prove" for each statement. He was cautioned to use the simplest possible labeling and to show in the figure only those qualities specified in the problem statement. (Many subjects in the earlier OP-RP comparisons were confused by their own drawing and labeling of the figures.) He was given illustrations showing how much simpler it was to work with an angle "3" than with an angle "ABC" (see Appendix N for example), and how misleading it could be to draw an isosceles triangle when the problem statement did not identify the triangle as isosceles.

In summary, the booklet attempted to help the subject obtain as much useful information as possible from the verbal problem statement and from the symbolic representation of that statement. It also gave him experience in constructing geometric figures according to specifications.

Booklet II: Finding the contents of a geometry statement. Once the subject had identified what was given and what he was to prove, he had to find Ideas (theorems, axioms, postulates, and definitions) that could serve as links in a logical chain connecting these terminal points in the proof. This booklet was intended to teach the subject to identify the essential contents of an Idea, so that he would know whether that Idea would help in a proof.

The subject was first told that, in order to identify the contents of a geometry Idea, he must "strip it down to its bare essentials." That is, he had to look for two things: (1) what kinds of geometric elements (e.g., lines, angles, triangles) were involved, and (2) how the elements were related (if they were lines or angles, were they equal or unequal; if triangles, were they congruent or similar?). Since every geometry Idea has both a hypothesis (the "if" part) and a conclusion (the "then" part), the subject had to identify the contents of each part independently. He was given a series of illustrations showing him how to state the contents of the hypotheses and conclusions of different Ideas. For example, the

hypothesis of the geometric statement, "If two angles of a triangle are equal, then the opposite sides are equal," had the essential contents, "angle equals angle," and the conclusion had the contents, "line equals line."

The remainder of the booklet gave the subject practice in identifying the contents of a variety of geometry ideas.

Booklet III: Generating a list of Ideas with specified contents. To add a link to a theorem proof, the subject first had to determine the contents which the link required (as taught in Booklet II), and then to identify all the Ideas that had these contents. Booklet III was intended to teach him to generate a list of Ideas having any specified contents. For example, he had to be able to list all previously presented Ideas having the conclusion contents, "angle equals angle."

The booklet started by telling the subject that his success in proving theorems would depend in large measure on his ability to remember all the Ideas that might be applied to the problem, and pointed out that the one Idea he overlooked might be the key to the solution. It then presented a catalog of all geometry Ideas that had previously been studied (see Appendix O), including those covered in first-semester geometry. In this catalog, Ideas were grouped by conclusion contents (i.e., all Ideas having the contents, "angle equals angle," were in one group, all those having the contents, "line equals line," were in another group, etc.). The use of the catalog was explained, but the subject was told that he must rely on the catalog as little as possible, since eventually he must be able to generate his own complete lists of Ideas without reference to the catalog.

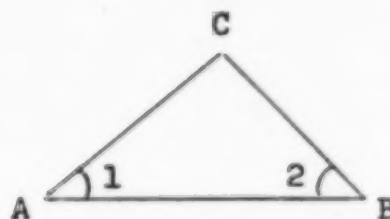
He was then given a series of conclusion contents, and for each set of contents he was required to list as many corresponding Ideas as he could recall. After he had generated each list he was told to check the list against the catalog so that he could correct his errors and omissions.

An Idea Wheel was prepared as a further aid to the subject's memory. The Idea Wheel was essentially a compact version of the familiar flip card. It consisted of three disks, one on top of the other, and all joined by a common axis. The two outer disks were in a fixed position relative to each other, but the inner disk could be rotated by means of a tab. One outer disk had a small slit, labeled "Conclusion Content"; the other had a large opening labeled "Idea List." These openings revealed materials printed on each side of the inner disk. The subject moved the tab until a set of conclusion contents appeared in the small slit. After he had tried to recall all the previous Ideas having those contents, he could turn the Idea Wheel over and read the correct list in the large aperture. He was given practice in using the wheel, and told to continue his practice until he could rely on his own memory to produce accurate and complete lists of Ideas.

Booklet IV: Applying an Idea in a given problem. After the subject had identified the necessary contents for a step in the proof, and after he had generated a list of Ideas having those contents, he had to examine the list further to see which Ideas were most applicable to the particular theorem. In Booklet IV, he was told that he could determine the probable applicability of a given Idea by comparing the elements (angles, lines, etc.) of that Idea with the elements in the figure. For example, the Idea's conclusion contents might be, "one line greater than second line." To apply the Idea, the subject had to look at the sketch or figure and ask, "What specific line in the sketch is larger than what second line?" The answer to this question was called the "Application of the Idea." If there were no elements in the sketch corresponding to the elements mentioned in the Idea, then the Idea was probably not applicable to the theorem proof.

The booklet gave practice in classifying different Ideas as applicable or not applicable to a given theorem. It also told the subject how to label the Idea elements to correspond to the sketch. In the sketch below, for example, the Application of the Idea contents, "angle equals angle," would be, " $\angle 1 = \angle 2$."

Given: Isosceles $\triangle ABC$



Booklet V: Using the subskills to prove theorems. To prove a theorem he had not previously seen, the subject had to assemble a new configuration of Ideas and sequence those Ideas to form a logically interconnecting chain leading from the hypothesis ("given") to the conclusion ("prove"). Booklet V was designed to help him construct such proofs by a synthesis of the more basic subskills he had previously learned.

The booklet started by summarizing the earlier subskills, and then introduced the concept of a "proof chain" in which individual Ideas formed the connecting links. To emphasize the chain analogy, the booklet depicted a physical chain, with each link in the chain labeled as a geometric statement. Next, it listed three general rules, or guidelines, that the subject should use in attempting to prove any theorem: (1) Get the problem on paper as simply as possible; (2) Try to work using the "given"; and (3) Try to work using the "prove."

A theorem was then presented in verbal form, and succeeding items demonstrated how the three guidelines might be applied to that theorem. One item illustrated the first major step, which included drawing a sketch, labeling the elements of the figure, and writing down what was given and what the subject was to prove.

Another item described how the second step, using the "given," should be applied to the theorem. It told the subject that he should always look for a direct consequence of the "given," which could be used as the next link from the "given" toward the "prove." For example, if it was given that a line bisected an angle, he could state by definition that the two resulting angles were equal to each other.

The next item told the subject that if he got stuck in working down from the "given" (i.e., if he could not see another direct consequence of the "given") he should switch his attention to the "prove" and attempt to add links from the "prove" back toward the "given." For example, if the "prove" was "angle a equals angle b," he was told to ask himself, "What would have to be true in the sketch for 'angle a equals angle b' to be true?" The item then discussed the logical reasoning that might be applied in the theorem being considered, in order to add a link preceding the "prove." Subsequent items expanded further on this method of adding links from the "prove" back toward the "given." To add links, the subject had to apply the subskills of identifying the contents of each link, generating a list of Ideas having the specified contents, and applying these Ideas to the theorem.

Preliminary Testing of Proof Booklets

The Proof Booklets were administered in individual tutorial fashion to three second-semester geometry students who had previously received the revised materials on inequalities (RP). At the conclusion of the study sessions, each subject was given the part of the criterion test requiring him to prove theorems. This preliminary administration was not intended to provide a rigorous assessment of the Proof Booklets, but simply to determine whether the subjects would even approach the theorem-proving skills that the new Booklets were designed to teach.

An informal analysis was made of the subject's performance, both in the program itself and on the criterion items. It was found that they performed satisfactorily on questions in the program, particularly in the first four Proof Booklets where they averaged about 85% correct responses. In the fifth booklet they answered about 65% of the questions correctly, but on the criterion items they averaged only about 42% correct responses.

Despite the crude nature of this preliminary test, two facts seemed obvious:
(1) the Proof Booklets were effectively teaching the four theorem-proving

subskills, and (2) the booklets still failed to teach the subjects how to synthesize a novel theorem proof completely; i.e., two out of the three subjects still could not start with a verbal statement of a non-trivial theorem and write a complete chain of steps forming the required proof.

Further analysis of the three subjects' responses, and of the skills required to prove the theorems in the criterion test, strongly suggested that the objectives represented by the criterion were unrealistic in view of the time limitations of the study. The basic problem was that in a non-trivial proof (i.e., one involving several steps), subskills simply cannot be applied in a mechanical, rule-following fashion; the subject must use judgment based on experience with a broad variety of previous theorems. Many students do not receive an adequate background of experience in first-semester geometry, and there was not time in this study to provide a sufficiently lengthy sequence of examples to build up this experience.

The following considerations may serve to illustrate some of the complexities of theorem-proving behavior:

- (1) The subject may find several geometry Ideas which could logically precede the terminal ("prove") step of the proof sequence. That is, each of these Ideas may have the appropriate contents and application. Several other Ideas may, in turn, be linked to each of those Ideas, so that a complex network of alternative branches may be generated back from the "prove." In a multi-step proof, the network may become too large for the subject to follow every possible path, and he must apply judgment in assigning priorities to these paths if he is to complete the proof in a reasonable period of time. This judgment depends in large measure on his past experience with other proofs.
- (2) The subject may find no path that connects logically with the initial ("given") statement, in which case he may have to alter the sketch by constructing new elements so that he can apply a new set of Ideas. Again, experience is his best aid in deciding when he must alter his sketch, and what elements he should construct.

It was estimated that several hundred additional items, possibly as many as a thousand, might be needed to teach all the necessary heuristics and strategies and to give the practice required to produce these complex theorem-proving behaviors in a high percentage (e.g., 80% or 90%) of the subjects. Since a programming effort of this magnitude was not possible in the time available, it was concluded that some of the program objectives must be revised. The revised objectives are reflected in the revised criterion test, described below.

Revision of the Criterion Test

In the revised criterion test, items were eliminated that required the subject to formulate a theorem proof completely. New items were added to test his ability to apply the individual theorem-proving subskills. In some of these items, for example, he was given a list of Ideas and asked to determine which of the Ideas could be applied to a specific theorem. Other items required him to generate lists of Ideas having certain contents.

Other new items presented proofs that lacked only one or two links, and required the subject to supply the missing link(s). (Appendix P shows one such item.) These items tested his ability to add individual steps in a proof, but did not expose him to the complexities of multi-step solutions involving many alternative paths.

The remainder of the criterion test was not significantly altered from its original form. The total possible score on the revised criterion test was 58 points.

In summary, the revised criterion test, while somewhat simplified, still required the application of facts and skills acquired in the program to a wide variety of problems. Many of these problems required transfer and generalizations to geometric configurations quite different from those presented in the program. In some of the test items, moreover, the subject had to synthesize multi-step solutions using several of the subskills in novel combinations. The major difference between the original and the revised criterion test was that the revised test placed greater emphasis on those skills specifically covered in the program, and less emphasis on judgmental skills that were largely dependent on prior experience and aptitude. (In this sense, the revision of the criterion test complemented the attempts to eliminate program gaps. Both actions were intended to make the program and the criterion test reflect the same program objectives.)

Revision of Proof Booklets

Based on the three subjects' responses to the original Proof Booklets, and on the changes made in the program objectives, a revised set of Proof Booklets was produced. Major revisions to the booklets are summarized below.

Booklets I and II. No substantial changes were made in the first two booklets. The performance of the three subjects indicated that these booklets satisfactorily produced the subskills for which the materials were designed.

Wording changes were introduced in some items, primarily to eliminate possible ambiguities concerning the kinds of responses that the subject was being asked to make in the original program. For example, he was

asked in one item to specify the conclusion of a theorem, and was intended to name the actual elements (angle "a," angle "b," etc.) in the conclusion. Two of the three subjects, however, stated the general contents of the conclusion ("one angle greater than second angle") without naming the elements. In the revised item, the subject was specifically told to label the elements by referring to the sketch provided in the problem.

Booklet III. The Idea Wheel proved cumbersome and slow to manipulate, and was eliminated in the revised booklet. One disadvantage of the Wheel was that since only a single set of contents could be seen at one time in the small aperture it was not possible to scan all the contents rapidly to find a specified set. Although this problem could have been remedied by the use of clear plastic for the outside disks, the Wheel was rejected as unnecessarily complex. In the revised booklet, the subject used the Idea Catalog as a reference source in place of the Idea Wheel.

"Idea Cards" were added to the revised booklet to wean subjects from dependence on the Idea Catalog. The front of each card contained a geometric statement (e.g., "The diagonals of a rectangle are equal"). The back of the card showed the contents of the statement's hypothesis and conclusions. The subject was required to read the statement on each card and, on a separate answer sheet, to rewrite the statement in "if...then" form. He then attempted to identify the essential contents of the statement and turned the card over to confirm his answer.

Another part of the booklet required the subject to sort all the cards according to their contents. This cataloging procedure was designed to develop his skill in generating his own lists of Ideas having specified contents.

Booklet IV. No substantial changes were made in this booklet.

Booklet V. This booklet was significantly influenced by the changes made in the program objectives, and was therefore drastically revised. The scope of the revised booklet was greatly reduced; it did not attempt to teach subjects how to prove theorems from scratch, but rather how to add one or two links to complete a proof. Even this more limited objective required the subject to use his subskills in a fairly complex manner, however, and many new items were added to the revised booklet to fill gaps in the sequence directing him in the integration of subskills. Thus the revised booklet contained more items than the original, more ambitiously conceived booklet (42 compared with 34).

The revised Booklet V, like the original, began with an analogy between proof sequences and physical chains. In the original booklet, however,

it was found that some subjects attempted to add illegitimate links--that is, statements that appeared to have the necessary contents, but were not true. An item was added to the revised booklet, therefore, reminding the subject that the only Ideas he could use as links were axioms and definitions (true without proof), and theorems that had already been proved.

The revised booklet then discussed the concept of linking in a much more detailed and operational fashion than was used in the original booklet. The subject was told that two statements were linked when the conclusion of the first statement was identical to the hypothesis of the second statement (see Appendix Q). Thus, to find a missing step, the subject had to find a statement whose hypothesis matched the conclusion of the preceding statement in the proof, or whose conclusion matched the hypothesis of the following statement.

In order to find the required statement, the subject had to apply the subskills he had been taught in the preceding booklets. He had to (1) identify the contents of the proof statement preceding or following the missing step, (2) generate a list of Ideas having those contents, and (3) attempt to apply each Idea to the theorem by labeling the Idea contents to correspond to the elements of the sketch.

He was then told that he must take certain precautions in the use of his subskills to find missing proof links (see Appendix R). This warning was designed to eliminate response errors found in the original booklet. The subject was told, for example, that he should not reject all other Ideas in his list as soon as he found one that could be applied to the theorem. Several Ideas might be applicable, and only later could he determine which one would actually complete the proof. He was also cautioned that his first list might be incomplete, so that none of the Ideas in the list would be found to apply. In this case he might have forgotten some of the appropriate Ideas, or he might have to construct new elements in the sketch so that other Ideas could be applied.

The remaining items in the booklet provided practice in using the theorem-proving subskills, individually and in combination, to fill missing lines in a proof (Appendix S shows one such example). Some of these items required the subject to find the statement that linked to a preceding step, and others to find a link to a subsequent step. Particular attention was given to the special problem of Ideas that had compound (multi-part) hypotheses or conclusions. The subject was told that, in such a proof, every part of the compound statement must be connected to its own logical chain.

In a number of the items, the subject was instructed to use the Idea Cards that were introduced in Booklet IV. He collected all of the cards having the appropriate hypothesis or conclusion contents, and then attempted to apply the statement on each card to the given theorem.

TUTORIAL SESSIONS WITH PROOF BOOKLETS

The revised inequality booklets (RP) and the revised Proof Booklets were given in individual tutorial fashion to three students from Santa Monica High School. They were second-semester geometry students who had not received any previous instruction in geometric inequalities and had not seen the earlier criterion test. After each subject completed the materials, he was given the revised criterion test.

The purpose of these sessions was to make further revisions in the Proof Booklets, and to determine the effectiveness of these booklets as indicated by performance on the criterion items requiring application of the theorem-proving subskills.

Only minor revisions were made in the Proof Booklets during the tutorial sessions, as the subjects showed little difficulty in answering questions in the booklets. (The poorest answered 81% of the questions correctly without tutorial aid, and the mean performance was 89%.) In Booklet III it was found unnecessary and excessively time-consuming for most subjects to identify the contents of every previous geometric Idea and to sort all these Ideas by contents. Consequently, some items in the booklet were changed so that the subject stopped the sorting process as soon as he had correctly placed three successive Ideas in each content category. Other minor program modifications included changes in item wording to eliminate possible ambiguities.

The three subjects who completed the Proof Booklets obtained a mean score of 84% correct responses on the new criterion test items dealing with the theorem-proving subskills (43 out of 51 possible responses). The poorest answered 76% of the items correctly.

No formal comparison was made between the original inequalities program (OP) and the final revised sequence (RP plus Proof Booklets). Such a comparison would not have been meaningful, because the program objectives and the criterion test had been modified. However, the theorem-proving portion of the criterion test was given to 12 subjects who were randomly selected from the same classes as the earlier four, but who had not received any programmed instruction. These subjects received a mean score of 19% (39 out of 204 possible), which was essentially at the chance level. It was clear, therefore, that the revised program, including the Proof Booklets, had produced an important increment in the subjects' ability to apply the theorem-proving subskills in a variety of geometry problems requiring different degrees of transfer.

SUMMARY OF PROGRAMMING PRINCIPLES

The remainder of this report is a summary of the most important observations made in this study regarding techniques of program design and improvement. Although the principles discussed here are derived from the empirical observations

made in revisions of the inequality materials and in the development and revision of the Proof Booklets, they represent an attempt in some cases to go beyond the direct evidence and to identify some of the major underlying factors that influence learning in the self-study situation.

Elimination of Irrelevant and Distracting Materials

In attempting to detect and remedy program inadequacies, primary attention is usually focused on portions of the program where new items must be added. The present study with geometry materials indicates, however, that some of the most significant program improvements may result from the elimination of certain items. Items may fail to contribute to the program objectives, and under some conditions may actually interfere with learning, either because they are irrelevant or because they provide superfluous over-training on skills previously acquired by the subjects.

If the extraneous items are completely unrelated to the program objectives, they may simply waste some of the subjects' study time. More serious effects may result, however, if these materials resemble, or are logically related to, the substantive and relevant portions of the program. In such cases, subjects may learn two different sets of responses to the two sets of stimuli (i.e., the relevant and the irrelevant items) and may fail to make the necessary discriminations among those items either in the program or in subsequent criterion performance. Maximum interference occurs when the irrelevant items require responses that compete with, or are incompatible with, the criterion responses.

In designing or revising a program, every item must be examined to determine whether it contributes toward some specific program objective, and items that do not contribute must be eliminated. In the more obvious cases, this examination may be based on a logical analysis of the relevance of the item to the program objectives. When an item's relevance is uncertain, empirical tests must be made to determine whether performance is improved or degraded by removal of the item. The exploratory tutorial approach described in this report provides an effective vehicle for such tests.

Addition of Materials to Fill Gaps

Probably the most common programming error is the failure to present all of the basic information on a given topic or program objective. (Failure to provide adequate practice or review of a topic is treated separately in the present analysis.) The resulting program gap causes errors on the topic concerned, both in the program and on the criterion test. Moreover, in a hierarchically structured content area such as geometry, the gaps cause errors on subsequent topics that are logically dependent on a complete mastery of the earlier materials.

The elimination of program gaps is dependent, first, on a thorough analysis of the sequence of behaviors required by each program objective. Next, the

information and subskills required for each of these behaviors must be identified. Entry skills must then be assessed so as to determine the training requirements, and finally a sequence of items must be prepared to fill those requirements.

Possible gaps in a program may be identified by a logical analysis of the existing items in relation to the program objectives. For each prerequisite subskill identified, the program may be examined to see whether there are items that specifically and explicitly cover that subskill.

Subjects' responses in a program may also help to identify gaps. Errors on a particular topic may suggest that some of the subskills relevant to that topic have not been covered. Items may be added, and the effects of these additions empirically determined in subsequent administrations of the program.

Addition of Preview Materials

The original geometry program contained almost no items telling the subject what topic or series of topics he would next be studying. The addition of such preview items was one of the most important program revisions in the present study.

It appears that the major function of the preview item is to reduce interference among topics by providing the subject with ready-made categories or labels under which he can store each new piece of information, principle, or example as it is presented in the program. These conceptual labels help the subject to associate each new piece of information with other pieces in the same category, and to make important discriminations between the new piece and other pieces filed in other categories. A preview covering several topics can also help him create meta-categories, by telling him what topics (e.g., subskills or specific facts) can be conceptually linked with or subsumed under other topics. Thus, preview items assist the subject in forming useful generalizations.

An item previewing a single topic (that is, telling the subject what topic he will next be studying) informs him that, until he receives further notice, every new item can be interpreted within a single conceptual framework different than that which applied to the preceding items, and that all of the items on this new topic will be closely related.

The above interpretation of the function of preview materials may be illustrated by the example, previously cited, of the preview preceding the Arithmetic Axioms of Inequality. The subject was told in the preview that he would be working on six different axioms, and that each of these axioms had a different application. Moreover, he was given a convenient way of identifying (labeling) each of the axioms. Thus, he knew that he should organize subsequent items under six different categories, and that information in one category should be

kept separate (discriminated) from information in another category. He was further aided in this categorization process by the preview item at the beginning of each of the Arithmetic Axioms. At the same time, however, the item previewing the six axioms told the subject that these axioms had certain characteristics in common. For example, all of them specified arithmetic manipulations (addition, subtraction, multiplication, and division) that could be used to combine two or more inequality and equality statements. Thus, the subject could form a meta-category of arithmetic axioms linked by these common (generalized) characteristics.

The success of a preview item depends largely on how clearly it defines the conceptual categories or topics that will be presented in subsequent items. In the case of the Arithmetic Axioms of Inequality this was fairly simple, since the basic arithmetic operations involved were already familiar to the subjects. It was more difficult to preview the theorem-proving subskills, since these subskills involved logical operations unfamiliar to some subjects. In preview situations of the latter type, it was sometimes helpful to use analogies to other skills or operations that the subject had previously mastered.

Addition of Review Materials

The review materials that were added to the geometry program aided learning in two ways. First, they strengthened individual subskills by giving the subjects additional practice. The original program had provided some immediate practice after the introduction of each new topic, but additional spaced review was necessary to maintain the response strength.

A second, and less obvious function of the review materials was to aid subjects in organizing a sequence of topics or a series of steps within a complex topic. In this sense, the review items complemented the preview items in helping the subject to relate topics to each other and to subsequent topics in the program, and in discriminating among the different topics.

Review materials intended to aid in the organization of related topics should present a synopsis of those topics, side by side on the same frame, so as to facilitate direct comparisons among the topics. If the topics are related in a hierarchical fashion, the logical structure of the dependency relationships should be shown as graphically as possible in the review.

The scope and the placement of the reviews are quite critical, and must be planned for a specific schedule of program administration. A review sequence that is appropriate for a one-day interval between study sessions will probably not be adequate for a one-week interval.

Addition of Branching Capabilities

In a hierarchically structured content area such as geometry, a subject's ability to achieve the program objectives is seriously degraded if he advances

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from one topic to the next without having mastered the earlier skills. His performance is weakened not only on the immediate topic, but on all subsequent topics that are logically dependent on the more basic subskills.

Under normal conditions (i.e., with typical heterogeneity among subjects), a single sequence of items is not appropriate for all. A slow learner may require twice as much review on a given topic as another, brighter subject. The instruction may be adapted to individual needs by the use of branching procedures. One type of branching automatically sends the subject back for further review on a topic if his performance on that topic does not meet certain standards. Another type, found effective for the high-school students used in the present study, permits the subject to branch back to any previous topic on which he himself believes he needs further instruction.

These two branching procedures, when applied in the revised geometry program, helped to ensure that subjects had complete mastery of each topic before advancing to subsequent topics, and produced improved criterion performance.

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APPENDICES

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APPENDIX A

CRITERION TEST USED IN COMPARISON OF ORIGINAL AND REVISED
GEOMETRY PROGRAMS.

Each item is marked A, B or C,
according to level of transfer
(A represents minimal and C
represents maximal transfer.)

Transfer
Category

- A 1. If we know that the statements $r > u$ and $r < u$ are both false, why is it true that $r = u$?

1. Transitive axiom.
2. Logical order law.
3. Trichotomy axiom.
4. The whole is greater than a part.
5. $\neq + \neq$ gives $\neq$.

- A 2. ABC is a straight line. Why is $AC > BC$?



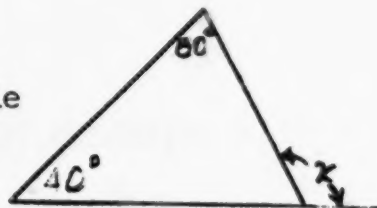
1. A quantity is greater than, less than, or equal to another quantity.
2. If $a > b$ and $b > c$, then $a > c$.
3. The whole is greater than any part.
4. It is farther from A to C than from B to C.

- A 3. When comparing any two sides in a triangle, the longer side is:

1. longer.
2. next to the shorter side.
3. formed by the larger angle.
4. opposite the larger angle.

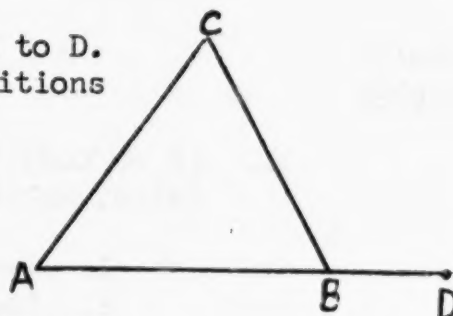
- A 4. Using the sketch, $\angle x = 120^\circ$ because:

1. the exterior angle of a triangle is greater than either remote interior angle.
2. whole = sum of its parts.
3. sum of $\angle$ s of a triangle = 180° .
4. exterior $\angle$ of a triangle = sum of the interior angles.
5. sum of remote interior $\angle$ s = the exterior angle.



- B 5. Given: ABC with side AB extended to D.
Under which of the following conditions would $\angle CBD < \angle ACB$?

1. If $\angle C$ is acute.
2. If $\angle CBA$ is obtuse.
3. If $\angle C$ is a right angle.
4. If $\angle A$ is acute.
5. None of the above conditions.



- A 6. If $\angle A > \angle B$, and $\angle C > \angle D$, why is $\angle A + \angle C > \angle B + \angle D$?

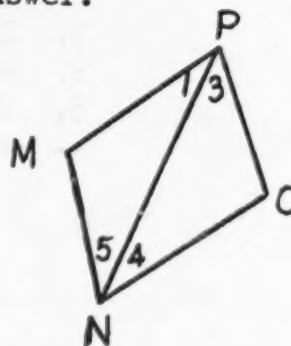
1. $\neq + \neq$ gives $\neq$ of same order.
2. $\neq + =$ gives $\neq$ of same order.
3. $= + \neq$ gives $\neq$ of same order.
4. $\neq + \neq$ gives $\neq$ of opposite order.
5. $= + \neq$ gives $\neq$ of opposite order.

- A 7. How long is the hypotenuse of a right triangle?

1. Always the longest side.
2. Sometimes the longest side.
3. Usually next to longest.
4. There is always a longer side.
5. There is not a definite answer.

- B 8. If $\angle MPO > \angle ONM$
and $\angle 1 = \angle 4$
then $\angle 3 ? \angle 5$

1. $>$
2. $<$
3. $=$
4. Can't say definitely.

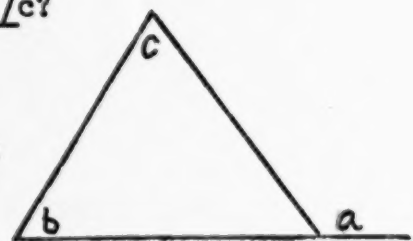


- B 9. In a right $\triangle$ with sides 9, 12, and 15, which side is the hypotenuse?

1. 9
2. 12
3. 15
4. It depends on the particular $\triangle$.

- A 10. Why is $\angle a$ never equal to either $\angle b$ or $\angle c$?

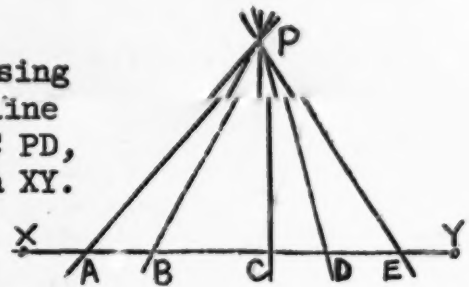
1. In a $\triangle$, the larger $\angle$ is opposite the larger side.
2. In a $\triangle$, the larger side is opposite the larger $\angle$.
3. The whole is $>$ any part.
4. Transitive axiom.
5. Exterior $\angle >$ either remote interior $\angle$.



- A 11. If equals are added to unequals, the sums are:

1. equal.
2. unequal.
3. equal in the same order.
4. unequal in the same order.
5. unequal in the opposite order.

- A 12. A "family" of lines has each one passing through a point P and intersecting line XY, thus forming segments PA, PB, PC, PD, and PE. PC forms a right angle with XY. Why is PC the shortest line segment formed in the family?



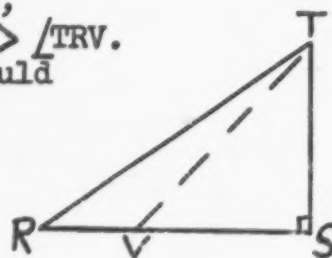
1. A straight line is the shortest distance between two points.
2. The whole is greater than any one part.
3. The perpendicular is the shortest distance from a point to a line.
4. The transitive axiom.
5. The trichotomy axiom.

- B 13. "If positive equals are $\left\{ \begin{array}{l} 1. \text{ added to} \\ 2. \text{ subtracted from} \\ 3. \text{ multiplied by} \\ 4. \text{ divided by} \end{array} \right\}$ positive

unequals, then the results are unequal in the same order." How many of the above statements are true? (Read, with #1, then #2, then #3, and finally #4.)

1. One statement is true.
2. Two statements are true.
3. Three statements are true.
4. Four statements are true.

- B 14. Given a right $\triangle RST$ as shown, and any point "V" on segment RS. If segment TV is drawn, it may be concluded that $TV < TR$ and $\angle TVS > \angle TRV$. Under which of the following conditions would the above conclusions no longer be true?



1. If $\triangle RTS$ is isosceles with base angles R and S equal.
2. If $\angle S$ is obtuse.
3. If point V lies on an extension of RS.
4. If $\angle T$ is a right angle and $TR > TS$.
5. If $\angle S$ is acute and $TR > TS$.

- A 15. If a quantity is less than a second, and the second is less than a third, then the:

1. first is less than the third.
2. first is less than the second.
3. second is greater than the third.
4. first is greater than the third.
5. third is less than the first.

- A 16. If, in a triangle, we know that one side is longer than another, what can we say about the angles opposite those sides?

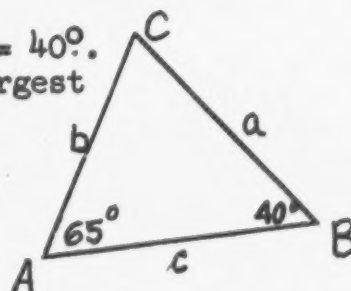
1. The larger angle is opposite the shorter side.
2. The smaller angle is opposite the shorter side.
3. The larger angle is opposite the smaller side.
4. There is nothing definite that may be concluded.

- A 17. If unequals are subtracted from equals, the results are:

1. equal, but in the opposite order.
2. unequal.
3. unequal in the same order.
4. unequal in the opposite order.
5. equal.

- B 18. In acute $\triangle ABC$, $\angle A = 65^\circ$ and $\angle B = 40^\circ$. Arrange the sides in order from largest to smallest.

1. abc
2. cba
3. cab
4. bac
5. bca



- B 19. What can we say about the size of the angles at the ends of the longest side of a triangle?

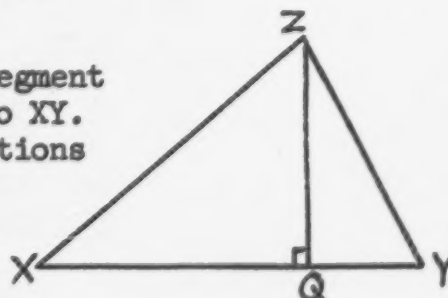
1. One of the angles could be a right angle.
2. One of the angles could be obtuse.
3. Both angles are equal.
4. Both angles must be acute.
5. We can't say anything definitely.

- A 20. Given the numbers A, B, C, and D are positive. If $A = B$ and $C > D$, then $A - C < B - D$, because:

1. $= - =$ gives $\neq$ of same order.
2. $= - \neq$ gives $\neq$ of opposite order.
3. $\neq - \neq$ gives $\neq$ of opposite order.
4. $= + \neq$ gives $\neq$ of opposite order.
5. $= - \neq$ gives $\neq$ of same order.

- B 21. Given: $\triangle XYZ$ and a point "Q" on segment XY such that ZQ is perpendicular to XY. Under which of the following conditions would ZY be shorter than ZQ?

1. If $\angle XZY$ is a right angle.
2. If $\angle Y$ is obtuse.
3. If points Y and Q coincide.
4. If $\angle Y$ is a right angle.
5. None of the above conditions.



A 22. If equals are subtracted from unequals, the results are:

1. equal.
2. unequal.
3. unequal in the same order.
4. unequal in the opposite order.

A 23. If $x > y$ and $a = b$, why is $x + a > y + b$?

1. Equals added to unequals gives unequals in the same order.
2. Unequals added to unequals gives unequals in the same order.
3. Transitive axiom.
4. Substitution.
5. Trichotomy axiom.

B 24. If $x > y$
and $m = n$
then $x/m ? y/n$

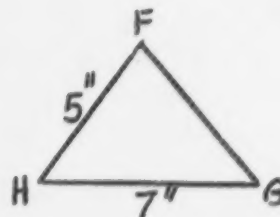
1. $>$
2. $<$
3. $=$
4. Can't say definitely.

B 25. $r > s$ and $l < m$. Combine these two statements by addition.

1. $r + l > m + s$.
2. $l + r < s + m$.
3. $m + r > s + l$.
4. $r + m < l + s$.
5. Answer not given.

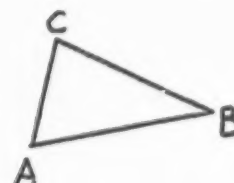
B 26. In order to form a $\triangle FGH$, what must FG be greater than if $FH = 5$ inches and $GH = 7$ inches?

1. 2 inches.
2. 5 inches.
3. 7 inches.
4. 12 inches.
5. Can't say definitely.



A 27. In $\triangle ABC$, $\angle A > \angle B$. Why is $BC > AC$?

1. The greater angle is opposite the greater side.
2. The greater side is opposite the greater angle.
3. The whole is greater than any one part.
4. The sum of the angles in a triangle is 180° .
5. An exterior angle is greater than either remote interior angle.

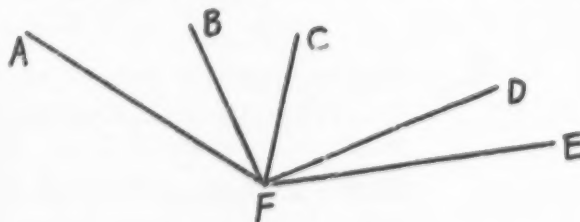


A 28. If unequals are added to unequals of the same order, the sums are:

1. equal in the same order.
2. unequal in the same order.
3. equal in the opposite order.
4. unequal in the opposite order.
5. it is not possible to say definitely.

B 29. Given: $\angle AFC = \angle CFE$, and $\angle DFE < \angle AFB$, then which of these statements is true?

1. $\angle BFC > \angle CFD$
2. $\angle CFB = \angle CFD$
3. $\angle CFD > \angle BFC$
4. $\angle AFD < \angle BFE$
5. $\angle DFC < \angle CFB$



B 30. If $m \neq y$ and m is not $< y$, then

1. $m > y$
2. $m < y$
3. $m = y$
4. Can't say definitely.

B 31. $\overline{A} \quad \overline{B} \quad \overline{C} \quad \overline{D} \quad \overline{E}$

If: $AB > BC$
 $BC > CD$
 $CD > DE$

then what is the relationship between AB and DE?

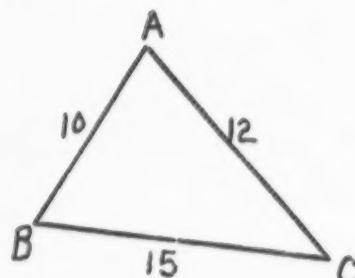
1. $AB > DE$
2. $AB < DE$
3. $AB = DE$
4. Can't say definitely.

B 32. If $\frac{m+n}{p} < 5$, what is the value of $m + n$?

1. $m + n = 5 + p$
2. $m + n > 5 - p$
3. $m + n < 5p$
4. $m + n < 5 - p$
5. Can't say definitely.

A 33. Using the information shown on the sketch, we conclude that $\angle C$ is the smallest angle in triangle ABC because:

1. the larger angle is opposite the larger side in a triangle.
2. the larger side is opposite the larger angle in a triangle.
3. any part is smaller than the whole.
4. any angle of a triangle is less than 180° .
5. an exterior angle of a triangle is greater than either remote interior angle.



B 34. The possible relationship between the populations of California and New York may be expressed in 4 ways:
 $C > N$; $C < N$; $C = N$; $C \neq N$
What is the greatest number of these statements that can be true at the same time?

1. One statement.
2. Two statements.
3. Three statements.
4. Four statements.

A 35. If: $s < q$ and $q < b$, then we conclude that $s < b$ by the reason:

1. substitution of inequalities.
2. trichotomy axiom of inequality.
3. transitive axiom of inequality.
4. both 1 and 3 above.
5. both 1 and 2 above.

A 36. If $AD + DC > AC$, why is $AD > AC - DC$?

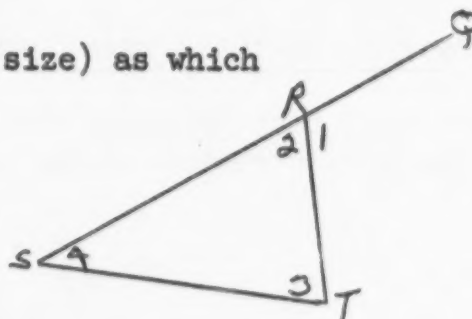
1. $= - \neq$ gives $\neq$ in the opposite order.
2. $\neq - =$ gives $\neq$ in the same order.
3. $\neq + \neq$ gives $\neq$ in the same order.
4. $\neq + =$ gives $\neq$ in the same order.
5. $\neq \times =$ gives $\neq$ in the same order.

B 37. You have learned a number of inequality axioms involving the addition, subtraction, multiplication, and division of equals and unequals. Some of these are true only for positive numbers, and some are true for all numbers. Which are true for positive numbers only?

1. Subtraction and multiplication.
2. Subtraction and division.
3. Multiplication and division.
4. Subtraction, multiplication, and division.
5. Addition, subtraction, multiplication, and division.

B 38. $\angle RTS$ has the same measure (size) as which of the following?

1. $\angle 2 + \angle 4$
2. $\angle 3 + \angle 4$
3. $\angle 1 - \angle 4$
4. $\angle 2 - \angle 4$
5. Answer not given.



A 39. What is another way of showing that $X < Y$?

1. $X < Y$
2. $Y > X$
3. $Y < X$
4. $Y = X$

- B 40. You have studied some axioms dealing with unequals (+, -, x, $\div$) equals. In which case or cases will a reversal of the words "equals - unequals" cause the order of the answer to change from "same order" to "opposite order" or vice versa?
1. Subtraction, multiplication.
 2. Subtraction, division.
 3. Multiplication, division.
 4. Subtraction, multiplication, division.
 5. Addition, subtraction, multiplication, division.
- B 41. Which of the following is an accurate and complete description about these mathematical statements: $a > b$; $b < a$.
1. The two statements are different.
 2. The two statements have the same meaning.
 3. They are in the same order.
 4. They have the same meaning but are in opposite order.
 5. They have a different meaning and are in opposite order.

IN ALL OF THE REMAINING TEST QUESTIONS:

Spaces have been provided for your answers. Place your answers neatly in the space as directed in the problem you are answering.

- C 42. Using the information on the sketch, with AB a straight line, solve for $\angle x$.

ANSWER:

$\angle x =$ _____



(show all necessary calculation in this box)

C

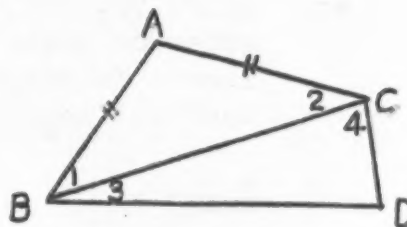
43. COMPLETE THE FOLLOWING PROOF:

GIVEN: $\triangle ABC$ is isosceles.
 $AB = AC$; $\angle 4 > \angle 3$

PROVE: $\angle ACD > \angle ABD$

PROOF: statements

reasons



1. $\angle 4 > \angle 3$	1. Given.
2. $\angle 2 = \angle 1$	2. _____
3. _____	3. _____
4. _____	4. _____

C

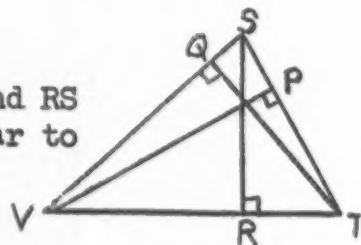
44. COMPLETE THE FOLLOWING PROOF:

GIVEN: $\triangle VST$ with altitudes QT , PV , and RS as shown. (An altitude is perpendicular to the side to which it is drawn.)

TO PROVE: $VS + ST + VT > VP + QT + SR$

PROOF: statements

reasons



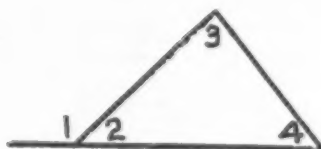
1. $\triangle VQT$, $\triangle VSP$, and $\triangle SRT$ are right triangles.	1. $\perp$'s form right $\angle$'s at P, Q, and R. Any $\triangle$ with a right $\angle$ is a right $\triangle$.
2. _____	2. _____
_____	_____

C

45. Below you see two steps of a plan for the proof that "The exterior angle of a triangle is equal to the sum of the remote interior angles." In the space provided, write in the next major step, and its reason.

Plan:

1. Show that $\angle 1 = 180^\circ - \angle 2$
(by supplementary $\angle$ s)
2. Show that $\angle 3 + \angle 4 = 180^\circ - \angle 2$
(by sum of the $\angle$ s of a $\Delta = 180^\circ$)
3. Show that _____
(by _____)

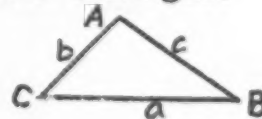


C

46. Complete the following formal proof by writing reasons for each of the statements given in the proof.

PROBLEM: Prove that one side of a triangle is always less than the semiperimeter of the triangle.

GIVEN: ΔABC with sides a , b , c .



PROVE: $a < \frac{a+b+c}{2}$

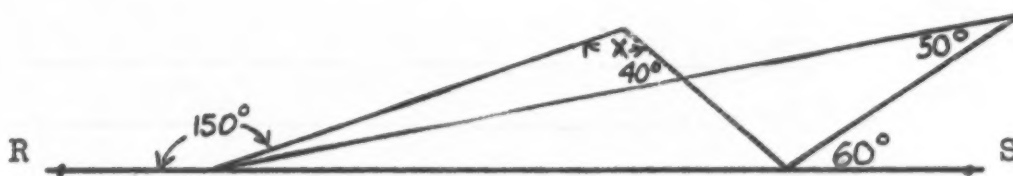
PROOF: statements

reasons

- | | |
|--|-----------------------|
| 1. ΔABC with sides a , b , c . | 1. Given information. |
| 2. $b + c > a$ | 2. _____ |
| 3. $\frac{b+c}{2} > \frac{a}{2}$ | 3. _____ |
| 4. $\frac{b+c}{2} + \frac{a}{2} > \frac{a}{2} + \frac{a}{2}$ | 4. _____ |
| 5. $\frac{a+b+c}{2} > a$ | 5. _____ |

C

47. Using the information on the sketch, and given that RS is a straight line, solve for $\angle x$.



ANSWER:

$$\angle x = \boxed{}^\circ$$

CALCULATIONS in the space below:

C

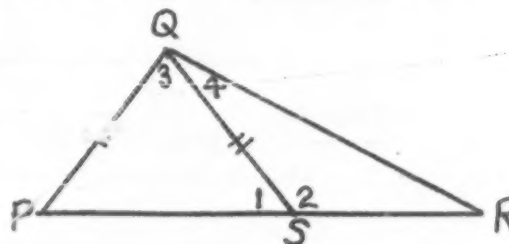
48. Complete the following problem as a formal proof:

GIVEN: $\triangle PQR$

$$PQ = QS$$

PROVE: $\angle P > \angle R$ PROOF: statements

reasons

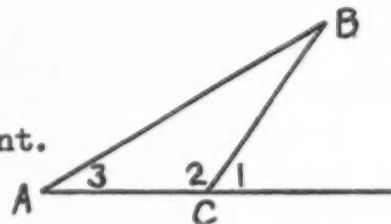
1. $\triangle PQR$; $PQ = QS$

1. Given.

2. _____

2. _____

C

49. GIVEN: $\triangle ABC$. Exterior $\angle 1$ is acute.PROVE: $AB > BC$. Give a reason for each statement in your argument.PROOF:

1. _____
2. _____

C

50. Complete the following problem as a formal proof. (Make a sketch and give a reason for each statement.)

Given any point P inside a $\triangle ABC$.
 Show that if each vertex is joined to P by a line segment, then the sum of these segments is greater than half the sum of the sides of the triangle.

GIVEN _____TO PROVE:

PROOF:

statements

reasons

- | | |
|----------|----------|
| 1. _____ | 1. _____ |
| _____ | _____ |
| _____ | _____ |

APPENDIX B

ITEM FROM REVISED PROGRAM, ILLUSTRATING THE USE OF INTRODUCTORY
MATERIALS WARNING THE SUBJECT OF CHANGE IN TOPIC.

We have already talked about subtracting an equality from an inequality. Now let's consider the reverse. If we subtract an inequality from an equality, we get an inequality in the opposite order from the original inequality.

APPENDIX C

ITEM FROM REVISED PROGRAM, ILLUSTRATING THE USE OF PREVIEWS
TO INFORM THE SUBJECT OF THE SEQUENCE OF COMING TOPICS

In the following booklets you will learn how to combine inequality statements with each other and with equality statements by means of simple arithmetic operations (addition, subtraction, multiplication and division). Each time you add an inequality statement to another, or subtract one from another, you produce a new inequality statement.

Look at the six rules below. They are called the Arithmetic Axioms of Inequality. Some of them look very similar to each other (and to the Arithmetic Axioms of Equality you learned earlier). However, each of these six new axioms has a different application from any other, and you will learn when and how it should be applied.

$\neq + = \rightarrow \neq$ of the same order

$\neq - = \rightarrow \neq$ of the same order

$\neq \times = \rightarrow \neq$ of the same order

$\neq \div = \rightarrow \neq$ of the same order

$\neq + \neq \rightarrow \neq$ of the same order

$= - \neq \rightarrow \neq$ of opposite order

2

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APPENDIX D

ITEM FROM REVISED PROGRAM, ILLUSTRATING THE USE OF SPACED REVIEW
ITEMS.

See if you remember this. What is the name of the following axiom? "A
given number when compared with another number is greater than, equal to, or
less than the other number."

APPENDIX E

ITEM FROM THE END OF A LESSON IN REVISED PROGRAM SUMMARIZING
EARLIER TOPICS.

Here is a summary of the inequality axioms you have studied in this booklet:

There are four axioms, one for each arithmetic process (+, -, x, ÷), which are identical except for the arithmetic process used. (Each is written with $\neq$ first and = second.)

$$\begin{array}{l} \neq + = \rightarrow \neq \text{ of same order} \\ \neq - = \rightarrow \neq \text{ of same order} \\ \neq x = \rightarrow \neq \text{ of same order} \\ \neq \div = \rightarrow \neq \text{ of same order} \end{array}$$

There are two other axioms, one for addition and one for subtraction. Try to remember how these axioms differ from the group shown above.

$$\begin{array}{l} \neq + \neq \rightarrow \neq \text{ of same order} \\ = - \neq \rightarrow \neq \text{ of opposite order} \end{array}$$

APPENDIX F

ITEM FROM THE BEGINNING OF A LESSON IN REVISED PROGRAM,
SUMMARIZING EARLIER TOPICS.

Before we introduce some new ideas, let's review briefly the topics covered in the previous lesson.

- (1) A straight line is the shortest distance between two points.
- (2) The length of one side of a triangle is less than the sum of the lengths of the other two sides.
- (3) The whole is equal to the sum of its parts and greater than any one part.
- (4) $a > b$ means "a is greater than b."
 $c < d$ means "c is smaller than d."

(Remember, the arrow points to the smaller number.)

$x \neq y$ means "x is unequal to y."

- (5) Trichotomy Axiom: A given number, when compared to another, is greater than, equal to, or less than the other given number.
 - (6) Two inequality statements are in the same order if the arrow symbols in the two statements point in the same direction.
-

APPENDIX G

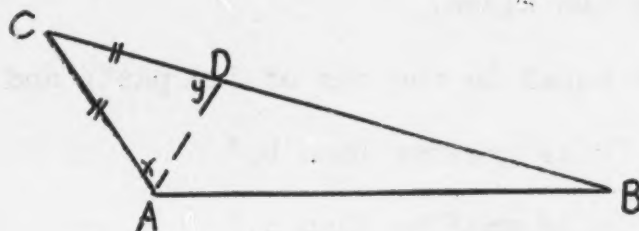
SEQUENCE OF ITEMS FROM REVISED PROGRAM, ILLUSTRATING THE CUMULATIVE
MODE OF PRESENTATION OF THEOREM PROOFS.

Now let's see if you remember the steps in the proof. The first step was
to select point D so that $CD =$ _____.

Given: In $\triangle ABC$, $BC > AC$

To prove: $\angle A > \angle B$

Proof:



statements

reasons

(1) In $\triangle ABC$, $BC > AC$.

(1) Given.

(2) Choose D so that $CD = ?$

APPENDIX G (cont'd)

CA

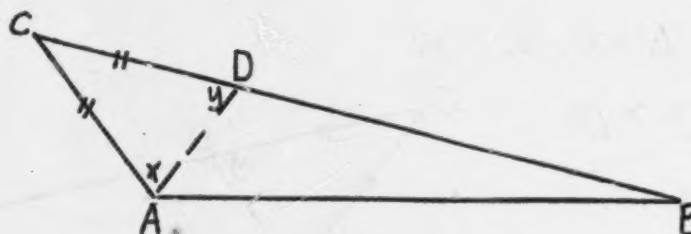
We have selected point D so that $CD = CA$. We then draw AD (two points determine a straight line).

We can see that $\angle Y > \angle B$ because _____.

Given: In $\triangle ABC$, $BC > AC$

To prove: $\angle A > \angle B$

Proof:

statementsreasons

1. In $\triangle ABC$, $BC > AC$.

1. Given.

2. Choose D so that $CD = CA$, and draw AD.

2. Two points determine a straight line.

3. $\angle Y > \angle B$

3. ?

APPENDIX G (cont'd)

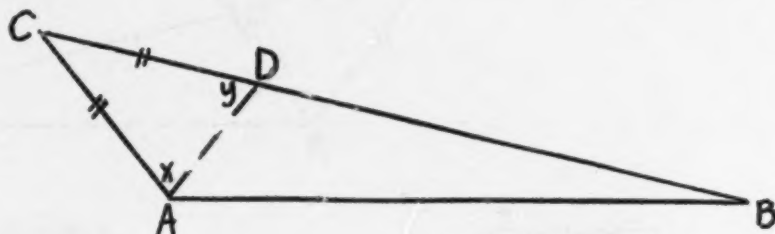
An exterior $\angle$ of a $\triangle$ is greater than either remote interior $\angle$. (Note that $\angle y$ is the exterior $\angle$ of $\triangle BDA$ and $\angle B$ is a remote interior $\angle$.)

Notice that we now have an angle ($\angle y$) that is greater than $\angle B$. But what we want to prove is that $\angle A$ is greater than $\angle B$. This means we should look for a relation between $\angle A$ and $\angle y$. To do this we will first relate $\angle y$ to $\angle x$, which is a part of $\angle A$.

Given: In $\triangle ABC$, $BC > AC$

To prove: $\angle A > \angle B$

Proof:



<u>statements</u>	<u>reasons</u>
1. In $\triangle ABC$, $BC > AC$.	1. Given.
2. Choose D so that $CD = CA$, and draw AD.	2. Two points determine a straight line.
3. $\angle y > \angle B$	3. An exterior $\angle$ of a $\triangle$ is greater than either remote interior $\angle$.

APPENDIX G (cont'd)

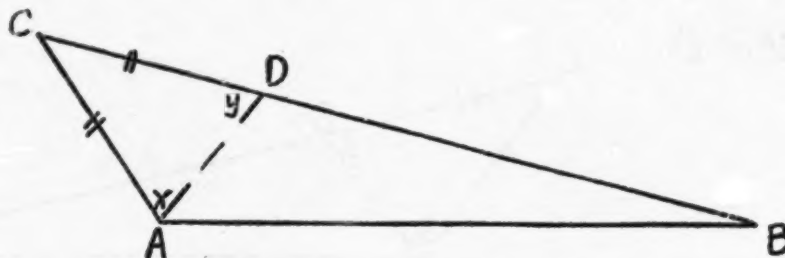
$\triangle ADC$ is an isosceles $\triangle$, because $CD = CA$.

$\angle x = \angle y$ because _____.

Given: In $\triangle ABC$, $BC > AC$

To Prove: $\angle A > \angle B$

Proof:



<u>statements</u>	<u>reason</u>
1. In $\triangle ABC$, $BC > AC$.	1. Given.
2. Choose D so that $CD = CA$, and draw AD.	2. Two points determine a straight line.
3. $\angle y > \angle B$	3. An exterior $\angle$ of a $\triangle$ is greater than either remote interior $\angle$.
4. $\angle x = \angle y$	4. ?

APPENDIX G (cont'd)

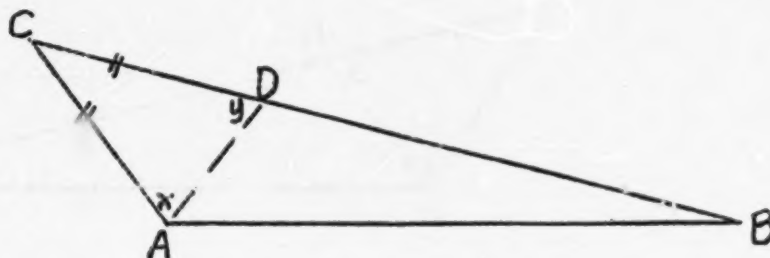
Base $\angle$ s of an isosceles $\triangle$ are = .

We have shown that $\angle y > \angle B$, and that $\angle x = \angle y$. By substitution of equals, $\angle x > \angle B$. But we can see in the figure that $\angle x$ is a part of $\angle A$, so $\angle A > \angle x$ because _____.

Given: In $\triangle ABC$, $BC > AC$

To prove: $\angle A > \angle B$

Proof:



<u>statements</u>	<u>reasons</u>
1. In $\triangle ABC$, $BC > AC$.	1. Given.
2. Choose D so that $CD = CA$, and draw AD.	2. Two points determine a straight line.
3. $\angle y > \angle B$	3. An exterior $\angle$ of a $\triangle$ is greater than either remote interior $\angle$.
4. $\angle x = \angle y$	4. Base $\angle$ s of an isosceles $\triangle$ are =.
5. $\angle x > \angle B$	5. Substitution of equals.
6. $\angle A > \angle x$	6. ?

APPENDIX G (cont'd)

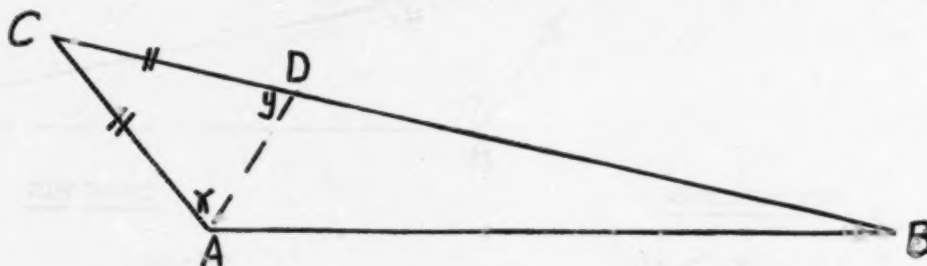
The whole is greater than any of its parts.

We have now shown that $\angle x > \angle B$, and that $\angle A > \angle x$. Therefore we can prove the final step, that $\angle A > \angle B$, because _____.

Given: In $\triangle ABC$, $BC > AC$

To prove: $\angle A > \angle B$

Proof:



<u>statements</u>	<u>reasons</u>
1. In $\triangle ABC$, $BC > AC$.	1. Given.
2. Choose D so that $CD = CA$, and draw AD.	2. Two points determine a straight line.
3. $\angle y > \angle B$	3. An exterior $\angle$ of a $\triangle$ is greater than either remote interior $\angle$.
4. $\angle x = \angle B$	4. Base $\angle$ s of an isosceles $\triangle$ are = .
5. $\angle x > \angle B$	5. Substitution of equals.
6. $\angle A > \angle x$	6. The whole $>$ any of its parts.
7. $\angle A > \angle B$	7. ?

APPENDIX G (cont'd)

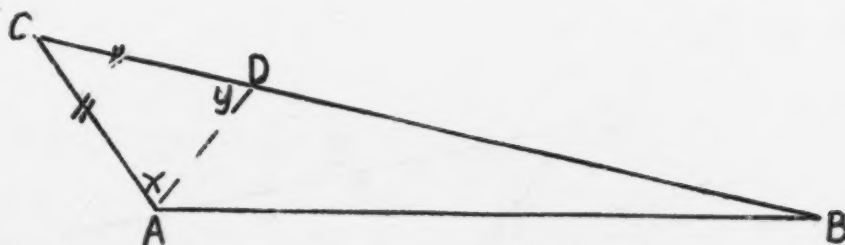
The "Transitive Axiom"
of unequals.

This gives us the final proof, as summarized below.

Given: In $\triangle ABC$, $BC > AC$

To prove: $\angle A > \angle B$

Proof:



<u>statements</u>	<u>reasons</u>
1. In $\triangle ABC$, $BC > AC$.	1. Given.
2. Choose D so that $CD = CA$, and draw AD.	2. Two points determine a straight line.
3. $\angle y > \angle B$	3. An exterior $\angle$ of a $\triangle$ is greater than either remote interior $\angle$.
4. $\angle x = \angle y$	4. Base $\angle$ s of an isosceles $\triangle$ are = .
5. $\angle x > \angle B$	5. Substitution of equals.
6. $\angle A > \angle x$	6. The whole $>$ any of its parts.
7. $\angle A > \angle B$	7. "Transitive Axiom" of unequals.

APPENDIX H

ITEM FROM REVISED PROGRAM, IN WHICH SUBJECT IS TOLD EXPLICITLY
HOW HE CAN DETERMINE WHETHER TWO INEQUALITY STATEMENTS ARE OF THE
SAME ORDER

(1)

$p > q$

$r > s$

(2)

$x > y$

$u < v$

The two statements in #1 above are in the same order because their arrows point in the same direction.

The two statements in #2 are not in the same order because their arrows point in the opposite direction.

Remember: nothing but the direction of the arrows tells you whether two statements are in the same order. You don't have to worry about what numbers "p" and "q" represent.

APPENDIX I

ITEMS FROM REVISED PROGRAM, IN WHICH SUBJECT IS EXPLICITLY
DIRECTED IN THE APPLICATION OF THE ADDITION PROCEDURE FOR
MORE THAN TWO INEQUALITY STATEMENTS.

If we add more than two inequality statements having the same order, we can still depend on the sums to be in that same order. To convince you that this really works, let's try "known" numbers that we can actually add. For example:

All three statements

$$3 > 2$$

have the same order.

$$6 > 1$$

$$\begin{array}{r} 4 > 3 \\ 13 > 6 \end{array}$$

As predicted the sums have the same order as the statements that were added.

APPENDIX I (contd)

The result ($13 > 6$) was obvious in the last example which had "known" members. The importance for our work is that even when we add inequality statements that contain some "unknown" numbers, we still know which sum is larger.

For example, suppose we know that $b > r$ and $x > 7$ and $a > m$. Since these three inequalities have the same order, we can conclude that $b + x + a$? .

APPENDIX J

ITEM FROM REVISED PROGRAM, ILLUSTRATING THE USE OF ITEMS THAT PRESENT INFORMATION BUT DO NOT REQUIRE RESPONSES.

Next write the hypothesis and the conclusion near the figure:

*Read and re-read the statment of the problem.

*Using the simplest names on your figure, tell yourself the given and what you will need to prove.

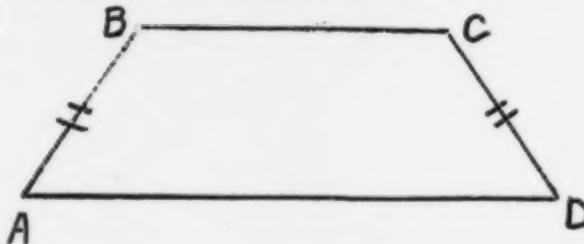
*Write these things near your sketch and label them as Given and Prove:

Here is an example:

Problem: Prove that the base $\angle$ s of an isosceles trapezoid are = .

Given: isos. trapezoid ABCD

Prove: $\angle A = \angle D$ and $\angle B = \angle C$



APPENDIX K

OPTIONAL REVIEW ITEM FROM REVISED PROGRAM.

The major ideas you have covered so far are listed below. You'll get more practice on most of these ideas later on in your lessons. However, if you feel confused on any of these topics at this point you can review the topic by turning back to the pages indicated in the listing below. When you are through reviewing a topic you can return to this page by flipping to the green tab.

- (1) A straight line is the shortest distance between two points (page 2).
 - (2) The length of one side of a triangle is less than the sum of the lengths of the other two sides (page 4).
 - (3) The whole is equal to the sum of its parts (page 9).
 - (4) $a > b$ means "a is greater than b" (page 11).
 $c < d$ means "c is less than d" (page 12).
 $x \neq y$ means "x is unequal to y" (page 19).
 - (5) Trichotomy Axiom: A given number, when compared to another, is greater than, equal to, or less than the other given number (page 26).
 - (6) The whole is greater than any one part (page 27).
 - (7) Two inequality statements are in the same order if the arrow symbols in the two statements point in the same direction (page 30a).
-

APPENDIX L

A PROGRAMMED BRANCH ITEM FROM REVISED PROGRAM.

$y < x$

Was your answer the same as that given in the box at the top of this page?

- (1) If so, skip to page 16.
 - (2) If not, go directly to page 15c.
-

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APPENDIX M

CRITERION SCORES FOR EXPERIMENTAL AND CONTROL SUBJECTS

Total Test		Part A		Part B		Part C	
Exper.	Control	Exper.	Control	Exper.	Control	Exper.	Control
35	33	18	18	13	11	6.5	4.3
40	29	15	16	11	8	5.0	5.3
30	36	19	16	14	14	6.8	5.5
35	30	12	14	9	11	3.8	5.3
24	27	19	11	11	15	5.5	1.3
37	30	18	15	12	10	5.5	5.0
36	27	20	13	12	12	4.5	1.8
36	24	11	13	11	11	2.0	0.3
25	45	17	19	12	18	6.0	8.3
40	42	15	18	13	16	2.3	8.3
31	30	19	14	16	12	5.3	3.8
38	32	19	17	12	11	3.8	3.5
32	31	18	13	15	13	6.5	5.0
38	39	15	16	15	18	5.0	5.0
32	37	20	17	16	14	8.3	5.5
34	34	16	15	13	12	5.8	7.3
35	36	17	16	13	13	3.8	6.5
44	29	16	13	12	12	4.0	3.8
36	39	19	16	14	15	4.8	7.5
40	30	15	13	11	14	5.8	3.0
44	25	15	10	12	10	3.8	5.3
40	39	14	19	8	13	2.0	7.3
32	33	20	16	17	12	5.8	4.5
47	28	16	15	13	8	5.5	4.8
35	21	19	11	13	9	9.0	1.0
41	30	16	12	11	11	3.8	6.5
31	33	19	16	15	12	6.5	5.0
41	20	17	14	12	5	5.8	0.5
35	35	20	18	18	12	9.0	5.3
43	12	15	6	12	6	5.0	0.3
24	31	17	15	15	12	7.5	3.8
31	37	20	16	18	15	6.0	6.0

APPENDIX N

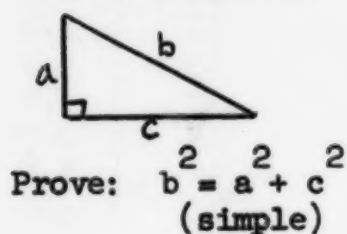
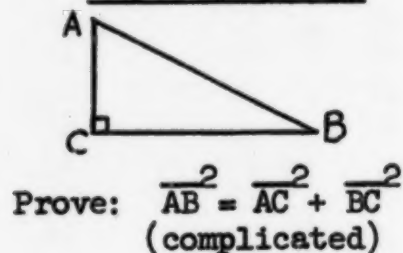
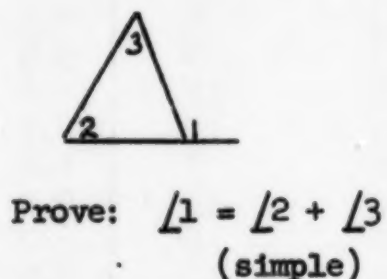
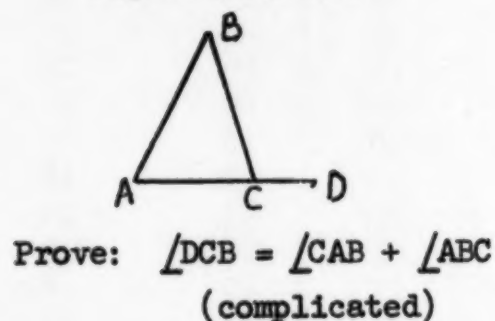
ITEM FROM REVISED PROGRAM, INSTRUCTING SUBJECT
TO USE SIMPLE LABELING OF FIGURES.

After you have drawn the figure for the problem, name the parts you will be working with:

*Don't label everything in the sketch. Name only lines and $\angle$ s needed for the Given and the Prove.

*Name by using single letters or numbers where possible.

Compare the figures below. The labeling in each figure #1 permits us to write a Prove that is simpler and thus clearer.

Problem 1:Figure #1(Good)Figure #2(Poor)Problem 2:Figure #1(Good)Figure #2(Poor)

APPENDIX O

CATALOG OF INEQUALITY IDEAS

SET	CONCLUSION CONTENTS	CARD NUMBER	IDEAS WITH THOSE CONCLUSION CONTENTS
D	Line $\neq$ line. (Line $>$ line or line $<$ line.)	7	Hypotenuse of right $\Delta >$ either leg.
		8	Straight line is shortest distance between two points.
		12	A $\perp$ is shortest distance from point to line.
		14	In a Δ , if $\angle > \angle$, then side $>$ side.
		42	Whole $>$ any of its parts.
		46	Transitive Axiom (if $a > b$ and $b > c$, then $a > c$).
		36	$\neq + \neq \rightarrow \neq$ of same order.
		39	$\neq \div = \rightarrow \neq$ of same order.
		43	$= - \neq \rightarrow \neq$ of opposite order.
		45	$\neq + = \rightarrow \neq$ of same order.
		48	$\neq \times = \rightarrow \neq$ of same order.
		49	$\neq - = \rightarrow \neq$ of same order.
E	$\angle \neq \angle$ ($\angle > \angle$ or $\angle < \angle$)	5	In a Δ , if side $>$ side, then $\angle > \angle$.
		10	In a Δ , exterior $\angle >$ remote interior $\angle$.
		28	Whole $>$ any of its parts.
		34	Transitive Axiom.
		21	$\neq - = \rightarrow \neq$ of same order.
		23	$\neq + \neq \rightarrow \neq$ of same order.
		24	$\neq + = \rightarrow \neq$ of same order.
		27	$\neq \times = \rightarrow \neq$ of same order.
F	Line $<$ sum of lines.	30	$\neq \div = \rightarrow \neq$ of same order.
		32	$= - \neq \rightarrow \neq$ of opposite order.
F	Line $<$ sum of lines.	1	In a Δ , side $<$ sum of other sides.
G	Line $\begin{pmatrix} > \\ \text{or} \\ = \\ \text{or} \\ < \end{pmatrix}$ line.	38	Trichotomy Axiom.
H	$\angle \begin{pmatrix} > \\ \text{or} \\ = \\ \text{or} \\ < \end{pmatrix} \angle$	33	Trichotomy Axiom.

APPENDIX O (cont'd)

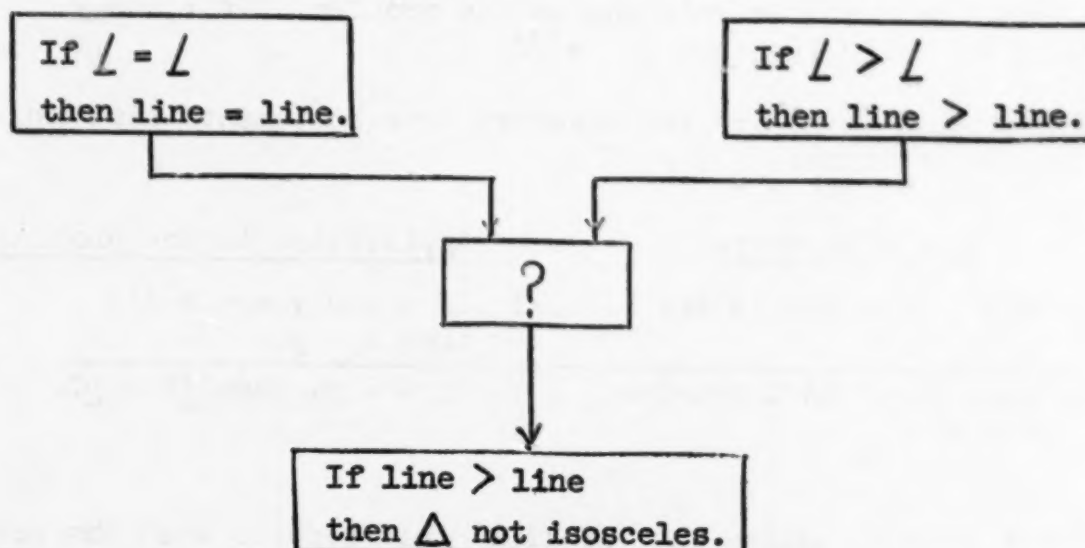
CATALOG OF EQUALITY IDEAS

SET	CONCLUSION CONTENTS	CARD NUMBER	IDEAS WITH THOSE CONCLUSION CONTENTS
I	Line = Line.	2	In a Δ , if $\angle = \angle$, then side = side.
		4	Radii of same circle are =.
		37	A bisector $\div$ into two = parts.
		40	$= \div = \rightarrow = .$
		41	$= - = \rightarrow = .$
		44	$= + = \rightarrow = .$
		47	$= \times = \rightarrow = .$
		50	Transitive Axiom (if $a = b$ and $b = c$ then $a = c$).
J	$\angle = \angle$	3	All straight $\angle$ s are = .
		6	All right $\angle$ s are = .
		9	In a Δ , if side = side, then $\angle = \angle$. } same
		9	Base $\angle$ s of an isosceles Δ are = . } idea
		13	Vertical $\angle$ s are = .
		15	An equilateral Δ is equiangular.
		16	Supplements of $\angle$ s are = .
		19	Complements of $\angle$ s are = .
		26	A bisector $\div$ into two = parts.
		22	$= - = \rightarrow = .$
		25	$= + = \rightarrow = .$
K	Line = sum of lines.	29	$= \times = \rightarrow = .$
		31	$= \div = \rightarrow = .$
		51	Transitive Axiom.
K	Line = sum of lines.	11	Perimeter of polygon = sum of sides.
		35	Whole = sum of its parts.
L	$\angle = \text{sum of } \angle\text{s.}$	17	In a Δ , exterior $\angle = \text{sum of remote interior } \angle\text{s.}$
		18	One straight $\angle$ (180°) = sum of interior $\angle\text{s of } \Delta$.
		20	Whole = sum of its parts.

APPENDIX P

ITEM FROM REVISED CRITERION TEST, REQUIRING SUBJECT
TO FILL MISSING LINKS IN A THEOREM PROOF.

Supply a statement that would satisfy the "linking" requirement for the
portion of the proof shown to the right.



APPENDIX Q

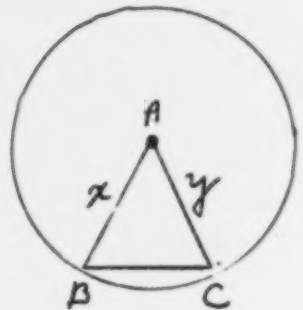
ITEM FROM REVISED PROOF BOOKLETS,
DEFINING AND ILLUSTRATING THE CONCEPT OF "LINKING" IN A PROOF.

Since the statement is not on the list, there is no point in trying to apply it to the problem. So we must look further for true statements that will connect the hypothesis and conclusion.

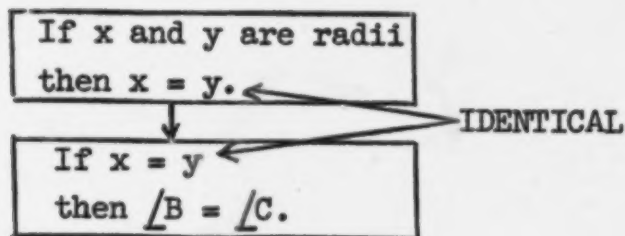
Let's look again at the solution to the problem, "If x and y are radii of a circle, then $\angle B = \angle C$."

In your catalog there are two accepted true statements that can help in the solution:

	<u>True Statements</u>	<u>Application in the problem</u>
Found in your catalog	1. Radii of a circle are =.	1. If x and y are radii then $x = y$.
	2. Base $\angle$ s of an isosceles are =.	2. If $x = y$, then $\angle B = \angle C$.



Most important of all, notice the "linking" that results when the conclusion of the first statement is identical to the hypothesis of the second statement.



APPENDIX R

ITEM FROM REVISED PROOF BOOKLETS, CAUTIONING SUBJECT IN
HIS APPLICATION OF THE THEOREM-PROVING SUBSKILLS.

You have been presented with a strategy for finding a missing step in a proof.

1. Examine the adjacent step for a "link" between the hypothesis of one and the conclusion of the other. Collect all statements that have this link.
2. Try to apply each of these possible statements (with the figure as an aid) rejecting any that do not seem to apply.

Before using the second principle you must be aware of two dangers. Suppose you used the first principle and developed a list of four possible statements, then you found that the first statement seemed to apply to the problem. Would you throw away the other three? Let's hope not, because your job at this point is to find every statement that might make a correct link.

The second danger can also lead to trouble. If your list of possible statements contained four statements and you already rejected the first three, would you automatically use the last? To do so could be an error. Maybe your list of possible statements was incomplete. (This can be avoided by careful use of the catalog.) It is also possible that none of the statements apply unless some lines are added to the figure.

Before turning the page, compare the restatement of strategy below with the one at the top of this page.

1. Examine the next step for a "link" between the hypothesis of one and the conclusion of the other. Very carefully collect all statements that have this link.
 2. Try to apply every one of these possible statements, rejecting those that do not seem to apply. If none seem to apply, try the list again, this time adding lines to the figure.
-

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APPENDIX S

ITEM FROM REVISED PROGRAM, GIVING PRACTICE IN USING
THE THEOREM-PROVING SUBSKILLS.

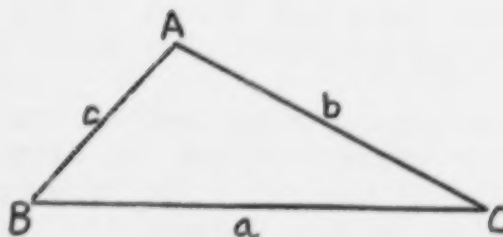
-
1. Whole $\angle$ $>$ its $\angle$ part. ($\angle B > \angle C$).
 2. In a Δ , larger $\angle$ is opposite larger side. ($\angle B > \angle C$.)
 3. Exterior $\angle$ of Δ $>$ remote interior $\angle$. ($\angle B > \angle C$.)

Test each of the statements above to see if they apply reasonably to the problem.

If - - -
then - - -



If $\angle B > \angle C$,
then ΔABC is not equilateral.



After you reject any unreasonable statements, list those that remain
(list by number).

?